

FOURIER MULTIPLIERS FOR WEIGHTED L^2 SPACES WITH LÉVY-KHINCHIN-SCHOENBERG WEIGHTS

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ABSTRACT. We present a class of weight functions w on the circle \mathbb{T} , called Lévy-Khinchin-Schoenberg (LKS) weights, for which we are able to completely characterize (in terms of a capacity inequality) all Fourier multipliers for the weighted space $L^2(\mathbb{T}, w)$. We show that the multiplier algebra is nontrivial if and only if $1/w \in L^1(\mathbb{T})$, and in this case multipliers satisfy the Spectral Localization Property (no “hidden spectrum”). On the other hand, the Muckenhoupt (A_2) condition responsible for the basis property of exponentials (e^{ikx}) is more or less independent of the Spectral Localization Property and LKS requirements. Some more complicated compositions of LKS weights are considered as well.

1. INTRODUCTION: FOURIER-HADAMARD MULTIPLIERS AND THE SPECTRAL LOCALIZATION PROPERTY (SLP)

Given a (nonnegative) finite Borel measure μ on the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, we define Fourier-Hadamard multipliers for the space $L^p(\mathbb{T}, \mu)$, $1 \leq p \leq \infty$, as

sequences of complex numbers $(\lambda_n)_{n \in \mathbb{Z}}$ such that the map
 $T: e^{inx} \mapsto \lambda_n e^{inx} \text{ (} n \in \mathbb{Z} \text{) extends to a bounded linear operator on } L^p(\mathbb{T}, \mu).$

The mapping T defined by a multiplier $(\lambda_n)_{n \in \mathbb{Z}}$ is also called a multiplier and denoted by $T = T_\lambda$. The set of all $L^p(\mathbb{T}, \mu)$ -multipliers endowed with the obvious operator (multiplier) norm is a unital Banach algebra of sequences on \mathbb{Z} denoted by $\text{Mult}(L^p(\mu))$. If μ is absolutely continuous with respect to Lebesgue measure m , and $\mu = wm$, we denote the corresponding class of multipliers by $\text{Mult}(L^p(w))$. It is clear that $(\lambda_n)_{n \in \mathbb{Z}} \in \text{Mult}(L^p(\mu)) \Rightarrow \sup_{n \in \mathbb{Z}} |\lambda_n| \leq \|T_\lambda\| < \infty$, so that always

$$\text{Mult}(L^p(\mu)) \subset l^\infty(\mathbb{Z}).$$

Despite the fact that multipliers play an important role in Fourier analysis, the only cases we know where the algebra $\text{Mult}(L^p(\mu))$ has been characterized explicitly are $\mu = m$ and $p = 1, 2, \infty$:

$$\text{Mult}(L^2(m)) = l^\infty(\mathbb{Z}), \quad \text{Mult}(L^1(m)) = \text{Mult}(L^\infty(m)) = \mathcal{FM}(\mathbb{T}),$$

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$\mathcal{F}\mu = (\hat{\mu}(n))_{n \in \mathbb{Z}}$ being the Fourier transform on \mathbb{T} , and $\mathcal{M}(\mathbb{T})$ the space of all complex Borel measures on \mathbb{T} .

Clearly, $\lambda = (\lambda_n)_{n \in \mathbb{Z}} \in \text{Mult}(L^p(\mu))$ if and only if $\lambda = \mathcal{F}k$, where k is a pseudo-measure on \mathbb{T} such that the corresponding convolution operator $T_\lambda f = k \star f$ is bounded: there exists a positive constant C for which

$$\|k \star f\|_{L^p(\mu)} \leq C \|f\|_{L^p(\mu)},$$

for all trigonometric polynomials f .

Many sufficient conditions are known for a sequence $(\lambda_n)_{n \in \mathbb{Z}}$ to be a multiplier (mostly for the “flat” case $\mu = m$, or for $\mu = wm$, where w satisfies the Muckenhoupt condition (A_p)), starting with the famous theorems of J. Marcinkiewicz, S. Mikhlin, L. Hörmander, S. Stechkin, followed by their improvements via the Littlewood-Paley theory, etc.; see [Duo2001], [Gra2008], [Tor1986].

However, the known results leave open many natural questions, in particular,

what does the spectrum of a multiplier T_λ , $\lambda = (\lambda_n)_{n \in \mathbb{Z}}$, look like?

This question is important for solving convolution equations, spectral theory of discrete operators of Schrödinger type, etc. Of course, obviously by definition the eigenvalues (λ_n) are in the spectrum $\sigma(T_\lambda)$ of T_λ , and a natural conjecture can be, *whether we have*

$$\sigma(T_\lambda) = \text{clos}\{\lambda_j : j \in \mathbb{Z}\},$$

for an arbitrary $T_\lambda \in \text{Mult}(L^p(\mu))$? If this is true for every $T_\lambda \in \text{Mult}(L^p(\mu))$, we say, following [Nik2009], that the *Spectral Localization Property* (SLP) holds. Clearly, the SLP is equivalent to the following *inverse closedness property*

$$(T_\lambda \in \text{Mult}(L^p(\mu)), |\lambda_j| \geq \delta > 0, \forall j) \Rightarrow T_\lambda^{-1} \text{ is bounded (i.e., is in } \text{Mult}(L^p(\mu)).$$

It is known that the algebras $\text{Mult}(L^p(m))$, $p \neq 2$, do not have the SLP. (For $p = 1$ and $\text{Mult}(L^1(m)) = \mathcal{FM}(\mathbb{T})$ this is the so-called *Wiener-Pitt-Shreider phenomenon*, see [GRS1960], and for $1 < p < \infty$, $p \neq 2$, its generalization to $L^p(m)$ spaces due to S. Igari and M. Zafran, see [GMcG1979].) In this paper, we give nontrivial examples of algebras $\text{Mult}(L^2(w))$ (w is not equivalent to a constant) satisfying the SLP.

It is also important to know whether there exists an estimate for $\|T_\lambda^{-1}\|$ in terms of the lower spectral parameter

$$\delta_T = \inf_j |\lambda_j(T)|.$$

The following quantity is responsible for such a property,

$$c_1(\delta) = c_1(\delta, \text{Mult}(L^p(\mu))) = \sup \left\{ \|T^{-1}\| : T \in \text{Mult}(L^p(\mu)), \|T\| \leq 1, \delta_T \geq \delta \right\},$$

where $0 < \delta \leq 1$.

It is known that for some function systems (say, for complex exponentials $e^{i\lambda x}$, $\lambda \in \sigma \subset \mathbb{C}$ in certain Banach spaces), even if the multiplier algebra is inverse closed, it does not imply that we automatically have a norm estimate for inverses (i.e., it may happen that $c_1(\delta) = \infty$ for some $\delta > 0$); see [Nik2009] and the references therein. However, for the multiplier algebras appearing in this paper, the situation is better: the inverse closedness yields an “automatic” norm estimate for inverses (i.e., $c_1(\delta) < \infty$, $\forall \delta > 0$); see, for instance, Lemma 2.2 below.

In the present paper, we limit ourselves to the Hilbert space case, $p = 2$, and $\mu = wm$ (except for a few general remarks). In fact, the (open) problem of the spectral localization property was the main motivation for the present study.

Recall that, in general, if a bounded operator $T: H \rightarrow H$ on a Hilbert space H has a Riesz (unconditional) basis (e_j) of eigenvectors, $Te_j = \lambda_j e_j$ ($j \in J$), then, of course, the spectral localization property holds: $\sigma(T) = \text{clos}\{\lambda_j : j \in J\}$. One could hope that if we replace “the Riesz basis” by “the (Schauder) basis”, then the SLP would still be true. At least, the SLP holds for multipliers defining the basis property: $(e_j)_{j \geq 1}$ is a (Schauder) basis if and only if every sequence (λ_j) of bounded variation $\sum_j |\lambda_j - \lambda_{j+1}| < \infty$ is a multiplier, and if such a sequence is separated from zero, $\inf_j |\lambda_j| > 0$, then the inverse $(1/\lambda_j)$ is again of bounded variation, and hence a multiplier.

However, in general, this is not the case: given a complex number α , $|\alpha| = 1$ (not a root of unity), there exists a Muckenhoupt weight $w \in (A_2)$ such that $(\alpha^j)_{j \in \mathbb{Z}} \in \text{Mult}(L^2(w))$ but $\sigma(T_\alpha) = \mathbb{D}$ (the closed unit disc), in particular $(1/\alpha^j)_{j \in \mathbb{Z}}$ is not a multiplier; see [Nik2009]. We show below (Theorem 5.13) that the existence of the hidden spectrum $\sigma(T_\lambda) \setminus \text{clos}\{\lambda_j : j \in \mathbb{Z}\}$ in such examples is caused by a kind of “forced holomorphic extension” of the function $j \mapsto \lambda_j$.

For the main class of weights w considered in this paper, namely, the “*Lévy-Khinchin-Schoenberg weights*” (LKS, for short) described below, we will see that the following alternative holds: either such a weight $w \in L^1(\mathbb{T})$ satisfies the integrability condition $1/w \in L^1(\mathbb{T})$, and then $\mathcal{S}_0 \subset \text{Mult}(L^2(w))$ (here \mathcal{S}_0 stands for finitely supported sequences), or $1/w \notin L^1(\mathbb{T})$, and then $\dim \text{Mult}(L^2(w)) < \infty$ (and, in fact, $\text{Mult}(L^2(w)) = \{\text{const}\}$ in the generic case); in both cases the SLP holds for $\text{Mult}(L^2(w))$.

The Muckenhoupt condition $w \in (A_2)$ (and consequently the fact that the exponentials $(e^{ijx})_{j \in \mathbb{Z}}$ form a Schauder basis in $L^2(w)$) plays no essential role for the SLP: we will see that LKS weights satisfying $1/w \in L^1(\mathbb{T})$ can obey ($w \in (A_2)$), or disobey ($w \notin (A_2)$) the Muckenhoupt condition, and still have the SLP.

Speaking informally, our main message regarding the SLP is the following. We say that a point $\zeta \in \mathbb{T}$ is a *singularity* of a weight w if there is no neighborhood V of ζ such that $0 < \inf_V w \leq \sup_V w < \infty$; then, our results show that

- the SLP holds for weights w having a finite set of singularities and “behaving well” (monotone, or slightly better, see Comment 3.10) at every singular point;
- the SLP may fail if w has infinitely many singularities (see Theorem 5.13).

In Section 2, we develop a kind of general scheme to treat the multipliers for “difference defined” Besov-Dirichlet spaces. Of course, it is largely inspired by the famous Beurling-Deny potential theory [BeD1958], [Den1970], but there are some new details for the case of the discrete group \mathbb{Z} that we consider. In particular, these spaces are defined by a symmetric matrix $\mathcal{C} = (c_{j,k})$, $c_{j,k} \geq 0$, in such a way that their basic properties hold for an arbitrary such matrix (in particular, the SLP), and more specific ones - for \mathcal{C} satisfying the so-called “non-splitting condition.” In our principal applications (Sections 3 and 4), \mathcal{C} is a Toeplitz matrix $(c_{|j-k|})$, and the most complete information is obtained for power-like sequences (c_n) (the Riesz potential spaces).

In Section 3, following P. Lévy and A. Khinchin (and many others, in particular I. Schoenberg, J. von Neumann, M. G. Krein, et al.) we introduce a class of remarkable weights for which we will be able to describe all multipliers.

In Section 4, we complete the program of Section 3, giving a capacity description of multipliers of $L^2(w)$ with a Lévy-Khinchin-Schoenberg (LKS) weight. We also discuss a simpler characterization of multipliers which does not involve capacities, for LKS weights w with quasi-metric property. In particular, this non-capacity characterization is valid for multipliers of Besov-Dirichlet spaces of fractional order which correspond to weights $w(e^{i\theta}) = |e^{i\theta} - 1|^\alpha$, $0 < \alpha < 1$. For such weights the SLP is a discrete analogue of its continuous counterpart due to Devinatz and Hirschman [DH1959] for multipliers on the group \mathbb{R} .

Section 5 is concerned with certain non-LKS weights w which can be represented as products, or sums of reciprocals, of LKS weights. Such weights, with a finite set of singularities $\zeta_k = e^{i\theta_k}$ ($k = 1, 2, \dots, N$) on \mathbb{T} are no longer associated with spaces of Besov-Dirichlet type. Nevertheless, we will show that the class of multipliers $\text{Mult}(L^2(w))$ permits a complete description in terms of embedding theorems similar to those of Sections 2-4, and has the SLP. In the special case

$$w(e^{i\theta}) = \sum_{k=1}^d a_k |e^{i\theta} - \zeta_k|^{-\alpha}, \quad a_k > 0, \quad 0 < \alpha < 1,$$

the characterization of $\text{Mult}(L^2(w))$ depends on the geometry of the points $\{\zeta_k\}$. (A similar characterization holds by duality for $w = \prod_{k=1}^d |e^{i\theta} - \zeta_k|^\alpha$.)

In particular, if d is a prime number, then either $\text{Mult}(L^2(w))$ coincides with $\text{Mult}(L^2(w_\alpha))$, where $w_\alpha = |e^{i\theta} - 1|^\alpha$, provided $\{\zeta_k\}$ are not the set of vertices of a regular polygon, or otherwise with $\text{Mult}(L^2(w_\alpha(e^{id\theta})))$ where $w_\alpha(e^{id\theta}) = |e^{id\theta} - 1|^\alpha$ is equivalent to an LKS weight with zeros at the roots of unity of order d . If d is not a prime number the answer is more complicated; it depends on the divisors of d and involves “aliases” of regular polygons (see Theorem 5.1).

For weights of this type with *infinitely* many singularities,

$$w(e^{i\theta}) = \sum_{k=1}^{\infty} a_k |e^{i\theta} - \zeta_k|^{-\alpha},$$

where $a_k > 0$, $\sum_{k=1}^{\infty} a_k < \infty$, and $0 < \alpha < 1$, it was shown in [Nik2009] that the SLP may actually fail.

In this case $w = w_{-\alpha} \star \nu$, where $\nu = \sum_{k=1}^{\infty} a_k \delta_{\zeta_k}$, and $\text{Mult}(L^2(\nu)) \subset \text{Mult}(L^2(\nu \star w_{-\alpha}))$. In Section 5, we complete these results of [Nik2009], by giving a description of $\text{Mult}(L^2(\nu))$ in order to show, as mentioned above, that the nature of the hidden spectrum of a multiplier $\lambda = (\lambda_j)$ lies in a “forced holomorphic extension” of the symbol $j \mapsto \lambda_j$, $j \in \mathbb{Z}$ (Theorem 5.13).

2. DISCRETE BESOV-DIRICHLET SPACES

In this section, we work with sequence spaces on \mathbb{Z} (having in mind $\mathcal{FL}^2(\mathbb{T}, w)$ with $1/w \in L^1(\mathbb{T})$, see Section 3 below). An obvious key observation is that multipliers of a “difference defined space” always obey the SLP (see Lemma 2.2 below). Given a matrix $\mathcal{C} = (c_{j,k})$, $c_{j,k} = c_{k,j} \geq 0$, $c_{j,j} = 0$ ($j, k \in \mathbb{Z}$) and an exponent p , $1 \leq p < \infty$, we define a (little discrete) *Besov-Dirichlet space* $\mathcal{B}_0^p(c_{j,k})$ on \mathbb{Z} in two steps: first, set

$$\mathcal{B}^p(c_{j,k}) = \left\{ x = (x_j)_{j \in \mathbb{Z}} : \|x\|^p = \|x\|_{\mathcal{B}^p(\mathcal{C})}^p = \sum_{j,k} c_{j,k} |x_j - x_k|^p < \infty \right\}$$

equipped with the corresponding (semi)norm $\|\cdot\|$. A special case important for applications (see Section 3) corresponds to $p = 2$ and $c_{j,k} = |j - k|^{-(1+\alpha)}$, $0 < \alpha < 1$. Explaining the terminology, recall a continuous prototype of this space, namely, the homogeneous Besov space $B_{\alpha}^{p,p}(\mathbb{R})$ corresponding to the norm

$$\|f\|^p = \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{|f(x) - f(y)|}{|x - y|^{\alpha}} \right)^p \frac{dx dy}{|x - y|},$$

as well as the Dirichlet space of holomorphic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ defined by

$$\begin{aligned} \|f\|^2 &= \sum_{n \geq 1} |\hat{f}(n)|^2 n = \int_{\mathbb{D}} |f'(z)|^2 \frac{dx dy}{\pi} \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{f(z) - f(\zeta)}{z - \zeta} \right|^2 dm(z) dm(\zeta), \end{aligned}$$

where m stands for normalized Lebesgue measure on \mathbb{T} . (The preceding expression for the Dirichlet norm is known as *Compton's formula* which goes back to the 1930s).

The celebrated Beurling-Deny theorem [BeD1958] shows that a Hilbert space seminorm $\|\cdot\|$ is of the form $\|\cdot\|_{\mathcal{B}^2(\mathcal{C})}$ for some matrix \mathcal{C} if and only if it is contractive for all Lipschitz maps $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ such that $|\Phi(z) - \Phi(\zeta)| \leq |z - \zeta|$

and $\Phi(0) = 0$: $\|\Phi(x)\| \leq \|x\|$ for every complex sequence $x = (x_j)_{j \in \mathbb{Z}}$, $\Phi(x) = (\Phi(x_j))_{j \in \mathbb{Z}}$.

In order to avoid unnecessary complications we often assume that

the matrix $\mathcal{C} = (c_{j,k})$ does not split (into two or more blocks):

if $A \subset \mathbb{Z}$ is such that $c_{j,k} = 0$ for every $j \in A$ and $k \in \mathbb{Z} \setminus A$ then either $A = \emptyset$ or $A = \mathbb{Z}$.

We denote by e_n the standard 0–1 sequence, $e_n = \mathcal{F}z^n = (\delta_{nj})_{j \in \mathbb{Z}}$, and observe that (for a non-splitting \mathcal{C}) $\|e_n\|^p = 2 \sum_j c_{j,n} > 0$ ($\forall n \in \mathbb{Z}$).

2.1. Lemma. *Assume \mathcal{C} is a non-splitting matrix. Then the following statements hold.*

(1) *One has $e_n \in \mathcal{B}^p(c_{j,k})$ if and only if $\sum_j c_{j,n} < \infty$.*

(2) *Assume $\sum_j c_{j,n} < \infty$ for every $n \in \mathbb{Z}$, and let*

$$\mathcal{S}_0 = \text{Lin}(e_n : n \in \mathbb{Z})$$

be a vector space of finitely supported sequences. Then $\|\cdot\|$ is a norm on \mathcal{S}_0 .

(3) *If one of the coordinate functionals $\varphi_n : (x_j)_{j \in \mathbb{Z}} \mapsto x_n$ is bounded on \mathcal{S}_0 (respectively, on $\mathcal{B}^p(c_{j,k})$), then all of them are bounded.*

Proof. Statement (1) is obvious. We prove (3) first. Without loss of generality, suppose φ_0 is bounded. The non-splitting hypothesis implies that for every $n \in \mathbb{Z}$ there exists a sequence $n_0 = 0, n_1, \dots, n_k = n$ (called the *chain* joining 0 and n) such that $c_{n_j, n_{j+1}} > 0$, for all $j = 0, \dots, k-1$. (Indeed, if A is the set of all $n \in \mathbb{Z}$ joinable to 0, then $c_{j,k} = 0$ for every $j \in A$ and $k \in \mathbb{Z} \setminus A$, and so $A = \mathbb{Z}$. Following [BeD1958], the existence of such a chain can also be called “ \mathcal{C} -connectedness of \mathbb{Z} .”) Hence, for every $x = (x_j)_{j \in \mathbb{Z}} \in \mathcal{B}^p$,

$$\begin{aligned} |x_n| &\leq |x_0| + \sum_{j=0}^{k-1} |x_{n_j} - x_{n_{j+1}}| \\ &\leq |\varphi_0(x)| + a \left(\sum_{j=0}^{k-1} c_{n_j, n_{j+1}} |x_{n_j} - x_{n_{j+1}}|^p \right)^{1/p} \leq A \|x\|_{\mathcal{B}^p}, \end{aligned}$$

and so $\|\varphi_n\| \leq A \leq \|\varphi_0\| + \sum_{j=0}^{k-1} (c_{n_j, n_{j+1}})^{-1/p}$.

To prove (2), notice that if $x \in \mathcal{S}_0$, $x_j = 0$ for $|j| > N$, and $\|x\| = 0$, the same reasoning as for (3) gives $x_k = 0$ for every $k \in \mathbb{Z}$ (fix a j with $|j| > N$ and join k to j by a chain). \square

Remark. For a general (non-zero) symmetric matrix $\mathcal{C} = (c_{j,k})_{j,k \in \mathbb{Z}}$, $c_{j,k} \geq 0$, instead of the non-splitting hypothesis, we can introduce in \mathbb{Z} an equivalence relation R saying that nRm if there exists a sequence of integers $j_1 = n, j_2, \dots, j_s = m$ such that $c_{j_k, j_{k+1}} > 0$ for $j = 1, \dots, s-1$ (nRn for every n by definition). Different cosets of this relation form a partition of \mathbb{Z} (finite or not), and the non-splitting hypothesis says that there is only one such “ \mathcal{C} -connected” component E_0 (equal to \mathbb{Z}).

These cosets are denoted by E_0, \dots, E_k, \dots . It is clear that $\|x\|^p = \sum_k \|x\chi_{E_k}\|^p$ for every $x \in \mathcal{S}_0$. The reasoning of Lemma 2.1 shows that $\|\cdot\|$ is a norm on $\mathcal{S}_0(E_k) = \text{Lin}(e_j : j \in E_k)$ if and only if E_k is infinite, and

$\|\cdot\|$ is a norm on \mathcal{S}_0 if and only if each E_k is infinite.

In this paper, the most important case is the *Toeplitz matrix case* $c_{j,k} = c_{|j-k|}$, where $c = (c_n)_{n \geq 0}$ is a given non-negative (non-zero) sequence (see Sections 2 and 3). In this case, the principal parameter is the number D defined by

$$D = D(\mathcal{C}) := \text{GCD}\{k \geq 1 : c_k > 0\}.$$

If p_1, \dots, p_m are such that $D(\mathcal{C}) = \text{GCD}\{p_j : j = 1, \dots, m\}$, then the \mathcal{C} -connected component E_0 containing $n = 0$ consists of the numbers $\sum_{j=1}^m p_j n_j$ ($n_j \in \mathbb{Z}$), and so is $D\mathbb{Z}$ (Bézout’s theorem); the other cosets are $j + D\mathbb{Z}$, $j = 1, \dots, D-1$. Therefore, for a non-zero Toeplitz matrix \mathcal{C} , $\|\cdot\|$ is always a norm on \mathcal{S}_0 .

In what follows, we always assume that $\mathcal{S}_0 \subset \mathcal{B}^p(\mathcal{C})$, and $x \mapsto \|x\|_{\mathcal{B}^p(\mathcal{C})}$ is a norm on \mathcal{S}_0 . We define a “little \mathcal{B}^p space” by

$$\mathcal{B}_0^p(c_{j,k}) = \text{span}_{\mathcal{B}^p(c_{j,k})}(e_n : n \in \mathbb{Z}),$$

where *span* means the “closed linear span” (or, better, the *completion* of $\mathcal{S}_0, \|\cdot\|$), and the *multipliers* of $\mathcal{B}_0^p(c_{j,k})$ by the following (standard) requirement:

$$\begin{aligned} & \text{Mult}(\mathcal{B}_0^p(c_{j,k})) = \\ & = \{(\lambda_j)_{j \in \mathbb{Z}} : T_\lambda : e_n \mapsto \lambda_n e_n, \forall n \in \mathbb{Z}, \text{ extends to a bounded map on } \mathcal{B}_0^p(c_{j,k})\}. \end{aligned}$$

It is clear that $\text{Mult}(\mathcal{B}_0^p(c_{j,k}))$ is a (commutative) unital Banach algebra.

2.2. Lemma. *The following statements hold.*

(1) Let $\lambda = (\lambda_j)_{j \in \mathbb{Z}}$ be a sequence of complex numbers. Then

$$\lambda \in \text{Mult}(\mathcal{B}_0^p(c_{j,k})) \Leftrightarrow \lambda \in l^\infty(\mathbb{Z}) \text{ and } \sum_k \left| x_k \right|^p \mu_k^p \leq C^p \|x\|_{\mathcal{B}^p}^p, \quad \forall x \in \mathcal{B}_0^p(c_{j,k}),$$

where $\mu_k^p = \mu_k^p(\lambda) =: \sum_j c_{j,k} \left| \lambda_j - \lambda_k \right|^p$, and C is a positive constant.

In particular, $\text{Mult}(\mathcal{B}_0^p(c_{j,k}))$ obeys the SLP, and

$$\|T_{1/\lambda}\| \leq \delta^{-2}(\|T_\lambda\| + \|\lambda\|_\infty) + \delta^{-1}, \text{ where } \delta = \inf_j |\lambda_j| > 0.$$

(2) If $C = C(\lambda)$ is the best possible constant in (1), then

$$C(\lambda) - \|\lambda\|_{l^\infty} \leq \|T_\lambda\| \leq C(\lambda) + \|\lambda\|_{l^\infty}.$$

Proof. If $\lambda \in \text{Mult}(\mathcal{B}_0^p(c_{j,k}))$, then clearly $\lambda \in l^\infty(\mathbb{Z})$, since λ_j are eigenvalues of a bounded operator. Hence, we can assume that $\lambda \in l^\infty(\mathbb{Z})$. We have

$$\lambda \in \text{Mult}(\mathcal{B}_0^p(c_{j,k})) \Leftrightarrow \sum_{j,k} c_{j,k} \left| \lambda_j x_j - \lambda_k x_k \right|^p \leq a^p \|x\|^p, \quad \forall x \in \mathcal{B}_0^p(c_{j,k}),$$

where $a = \|T_\lambda\|$ for (\Rightarrow) , or $\|T_\lambda\| \leq a$ for (\Leftarrow) . Let $x = (x_j)_{j \in \mathbb{Z}}$ be a finitely supported sequence. Then

$$\sum_{j,k} c_{j,k} \left| \lambda_j \right|^p \left| x_j - x_k \right|^p \leq \|\lambda\|_{l^\infty}^p \|x\|^p,$$

and hence

$$\begin{aligned} \|T_\lambda x\| &= \left(\sum_{j,k} c_{j,k} \left| \lambda_j x_j - \lambda_j x_k + \lambda_j x_k - \lambda_k x_k \right|^p \right)^{1/p} \\ &\leq \|\lambda\|_{l^\infty} \|x\| + \left(\sum_{j,k} c_{j,k} \left| \lambda_j - \lambda_k \right|^p \left| x_k \right|^p \right)^{1/p} \\ &= \|\lambda\|_{l^\infty} \|x\| + \left(\sum_k \left| x_k \right|^p \mu_k^p \right)^{1/p}. \end{aligned}$$

If the right-hand side inequality in (1) holds, we obtain

$$\|T_\lambda x\| \leq \left(\|\lambda\|_{l^\infty} + C \right) \|x\|,$$

which shows that $\lambda \in \text{Mult}(\mathcal{B}_0^p(c_{j,k}))$ and $\|T_\lambda\| \leq \|\lambda\|_{l^\infty} + C$. Conversely, if λ is a multiplier, we have as before,

$$\left(\sum_k \left| x_k \right|^p \mu_k^p \right)^{1/p} \leq \|\lambda\|_{l^\infty} \|x\| + \|T_\lambda x\|,$$

and so the right hand side inequality follows with $C \leq \|\lambda\|_{l^\infty} + \|T_\lambda\| \leq 2\|T_\lambda\|$.

The SLP follows from this description of multipliers by means of the embedding theorem: if $\lambda = (\lambda_j)_{j \in \mathbb{Z}} \in \text{Mult}(\mathcal{B}_0^p(c_{j,k}))$ and $\delta = \inf_j |\lambda_j| > 0$, then

$$\mu_k^p(1/\lambda) = \sum_j c_{j,k} \frac{|\lambda_j - \lambda_k|^p}{|\lambda_j \lambda_k|^p} \leq \frac{1}{\delta^{2p}} \mu_k^p(\lambda),$$

and hence $C(1/\lambda) \leq C(\lambda)/\delta^2$, $\|T_{1/\lambda}\| \leq C(1/\lambda) + \|1/\lambda\|_\infty \leq \delta^{-2}(\|T_\lambda\| + \|\lambda\|_\infty) + \delta^{-1}$.

It is clear that (2) is proved as well. \square

Remark. A slightly different estimate of $\|T_{1/\lambda}\|$ follows by a direct computation:

$$\begin{aligned} \|T_{1/\lambda}x\| &= \left(|x_0/\lambda_0|^p + \sum_{j,k} c_{j,k} \left| (x_j/\lambda_j) - (x_k/\lambda_k) \right|^p \right)^{1/p} \\ &= \left(|x_0/\lambda_0|^p + \sum_{j,k} c_{j,k} \left| \frac{\lambda_k(x_j - x_k) - \lambda_j(x_k - x_j) + \lambda_k x_k - \lambda_j x_j}{\lambda_j \lambda_k} \right|^p \right)^{1/p} \\ &\leq 2\delta^{-1} \|\lambda\|_\infty \|x\| + \delta^{-2} \|T_\lambda x\|, \end{aligned}$$

$$\text{so } \|T_{1/\lambda}\| \leq 2\delta^{-1} \|\lambda\|_\infty + \delta^{-2} \|T_\lambda\|.$$

Lemma 2.2 allows us to decide when the multiplier algebra is nontrivial, that is, $\text{Mult}(\mathcal{B}_0^p(c_{j,k})) \neq \{\text{const}\}$.

2.3. Lemma. *Under the hypothesis of non-splitting, the following statements are equivalent.*

- (1) $\mathcal{S}_0 \subset \text{Mult}(\mathcal{B}_0^p(c_{j,k}))$.
- (2) $\text{Mult}(\mathcal{B}_0^p(c_{j,k})) \neq \{\text{const}\}$.
- (3) All φ_n (see Lemma 2.1(3)) are bounded on \mathcal{S}_0 .
- (4) $(e_n)_{n \in \mathbb{Z}}$ is a minimal sequence in $\mathcal{B}_0^p(c_{j,k})$.

Proof. Clearly, (1) \Rightarrow (2) and (3) \Rightarrow (4) ($\varphi_n(e_k) = \delta_{nk}$); moreover, (4) \Rightarrow (3) since a sequence biorthogonal to (e_n) coincides with φ_n on \mathcal{S}_0 . Hence, (3) \Leftrightarrow (4), and consequently (4) \Rightarrow (1) is also easy: if $\sum_n |\lambda_n| \cdot \|\varphi_n\| \cdot \|e_n\| < \infty$, a multiplier $T_\lambda x = \sum_n \lambda_n \varphi_n(x) e_n$, $x \in \mathcal{S}_0$, is bounded.

Let us show (2) \Rightarrow (3): assume if φ_0 is not bounded, so are all φ_k (see Lemma 2.1(3)), and let $\lambda \in \text{Mult}(\mathcal{B}_0^p(c_{j,k}))$. By Lemma 2.2, there exists a constant C

such that $\sum_n \left| x_n \right|^p \mu_n^p \leq C^p \|x\|_{\mathcal{B}^p}^p$ for every $x \in \mathcal{S}_0$. Since $x_n = \varphi_n(x)$ this implies that $\mu_n^p(\lambda) := \sum_j c_{j,n} \left| \lambda_j - \lambda_n \right|^p = 0$ for all $n \in \mathbb{Z}$. Now, given $n \in \mathbb{Z}$, the block splitting hypothesis implies the existence of a chain $n_0 = 0, n_1, \dots, n_k = n$ such that $c_{n_j, n_{j+1}} > 0$ for all $j = 0, \dots, k-1$ (see arguments of Lemma 2.1), and hence $\left| \lambda_{n_j} - \lambda_{n_{j+1}} \right|^p = 0$ for all $0 \leq j \leq k-1$, which implies $\lambda_n = \lambda_0$, $\forall n \in \mathbb{Z}$. \square

Remark. In general (i.e., without the non-splitting hypothesis), properties (1), (3), and (4) are still equivalent. As to (2), using the mentioned above partition $\mathbb{Z} = \bigcup_{k \geq 0} E_k$ into a union of C -connected components, it is easy to see that if φ_n is bounded for some $n \in E_k$, then all $\varphi_j \in E_k$ are bounded, and this property is equivalent to the fact that $\text{Mult}(\mathcal{B}_0^p(c_{i,j})_{i,j \in E_k}) \neq \{\text{const}\}$. Conversely, if φ_n is unbounded for some $n \in E_k$ (and hence all $\varphi_j \in E_k$ are unbounded), then every multiplier $\lambda \in \text{Mult}(\mathcal{B}_0^p(c_{i,j}))$ is constant on E_k (with the same proof as above). In particular, if all $\varphi_j, j \in \mathbb{Z}$, are unbounded, then $\text{Mult}(\mathcal{B}_0^p(c_{i,j}))$ consists of all $l^\infty(\mathbb{Z})$ sequences constant on every $E_k, k = 0, 1, \dots$.

2.4. Example. Let $c_{k,k+1} = c_{k,k-1} = 1$ ($\forall k \in \mathbb{Z}$) and $c_{k,j} = 0$ for all other indices $(k, j) \in \mathbb{Z}^2$, so that

$$\|x\|^p = 2 \sum_{k \in \mathbb{Z}} \left| x_k - x_{k+1} \right|^p, \quad x \in \mathcal{S}_0.$$

Clearly, if $p > 1$, the functional φ_0 is not bounded (and it is, if $p = 1$), and by Lemma 2.3, $\text{Mult}(\mathcal{B}_0^p(c_{j,k})) = \{\text{const}\}$. For $p = 1$, $\mathcal{B}_0^1(c_{j,k})$ is the space of sequences tending to zero and of bounded variation; it is easy to see that $\text{Mult}(\mathcal{B}_0^1(c_{j,k})) = \mathcal{B}^1(c_{j,k})$, that is, the space of all sequences of bounded variation.

2.5. Comments. (1) On the definition of $\mathcal{B}_0^p(c_{j,k})$: what is the completion of \mathcal{S}_0 ? It is clear that if the coordinate functionals φ_n are bounded on \mathcal{S}_0 , then we can realize the completion of \mathcal{S}_0 (i.e., the space $\mathcal{B}_0^p(c_{j,k})$) as a sequence space. Much less clear is what happens beyond this condition. For the norm from Example 2.4, we will give a description of $\mathcal{B}_0^2(c_{j,k})$ in Section 3.

(2) In general, beyond the scope of Besov-Dirichlet spaces, the lack of minimality of a sequence $\mathcal{E} = (e_n)$ in a Banach space X does not prevent the multiplier algebra

$$\begin{aligned} \text{Mult}(\mathcal{E}) &= \{(\lambda_n) : e_n \mapsto \lambda_n e_n, \forall n, \\ &\text{extends to a bounded linear map on } \text{span}_X(\mathcal{E})\} \end{aligned}$$

to be nontrivial. A standard example is given by reproducing kernel sequences in a holomorphic space, say the Hardy space H^2 on the unit disc \mathbb{D} : whatever is a sequence $e_n = (1 - \overline{w_n}z)^{-1}$, $|w_n| < 1$, the sequence $\lambda_n = \varphi(\overline{w_n})$, where $\varphi \in H^\infty$, is in $\text{Mult}(\mathcal{E})$. (If (w_n) is not a Blaschke sequence, all multipliers are of that form.) Similar examples involve exponentials $e^{i\lambda x}$, $\text{Im}(\lambda) > 0$ in the space $L^2(0, \infty)$.

(3) It is not clear how to express the minimality property of Lemma 2.3 in terms of $c_{j,k}$. For an important partial case (the principal spaces of this paper), where $c_{j,k} = c_{|j-k|}$ and $c_n \geq 0$, $0 < \sum c_n < \infty$, we will give certain criteria in Section 3, in different forms. For example, the condition

$$\sum_{n \geq 1} \frac{1}{\sum_{k=1}^n c_k k^2 + n^2 \sum_{k > n} c_k} < \infty$$

is necessary, and

$$\sum_{n \geq 1} \frac{1}{\sum_{k=1}^n c_k k^2} < \infty$$

is sufficient for the minimality of (e_n) in the space $\mathcal{B}_0^2(c_{|j-k|})$. In particular, for $c_k = (1 + |k|)^{-(1+\alpha)}$ ($\alpha > 0$), the minimality holds if and only if $\alpha < 1$.

(4) One more property of spaces $\mathcal{B}_0^2(c_{j,k})$ that we will need in Section 3 is the uniqueness of the matrix $\mathcal{C} = (c_{j,k})$ in the definition of a Besov-Dirichlet norm, as stated in the following lemma.

2.6. Lemma. *Let $\mathcal{C} = (c_{j,k})$ and $\mathcal{C}' = (c'_{j,k})$ be two matrices satisfying the above conditions (i.e., non-splitting, and $e_n \in \mathcal{B}_0^2$, $\forall n \in \mathbb{Z}$; see Sec. 2.1), which define the same 2-norm: $\|x\|_{\mathcal{B}^2(\mathcal{C})} = \|x\|_{\mathcal{B}^2(\mathcal{C}')}$ for every $x \in \mathcal{S}_0$. Then $\mathcal{C} = \mathcal{C}'$.*

Proof. Let $x \in \mathcal{S}_0$ and $x_\theta = (x_k e^{ik\theta})$, where $\theta \in (-\pi, \pi)$. We have

$$\begin{aligned} \|x_\theta\|_{\mathcal{B}^2(\mathcal{C})}^2 &= \sum_{j,k} c_{j,k} |e^{ij\theta} x_j - e^{ik\theta} x_k|^2 \\ &= \sum_{j,k} c'_{j,k} |e^{ij\theta} x_j - e^{ik\theta} x_k|^2 = \|x_\theta\|_{\mathcal{B}^2(\mathcal{C}')}^2, \end{aligned}$$

and integrating in $\theta \in (-\pi, \pi)$, we get

$$\sum_{j,k} c_{j,k} (|x_j|^2 + |x_k|^2) = \sum_{j,k} c'_{j,k} (|x_j|^2 + |x_k|^2).$$

Using the preceding equality, we deduce

$$\sum_{j,k} c_{j,k} \text{Re}(e^{i(j-k)\theta} x_j \overline{x_k}) = \sum_{j,k} c'_{j,k} \text{Re}(e^{i(j-k)\theta} x_j \overline{x_k}).$$

Letting $\theta = 0$, we obtain

$$\sum_{j,k} (c_{j,k} - c'_{j,k}) x_j \bar{x}_k = \operatorname{Re} \left(\sum_{j,k} (c_{j,k} - c'_{j,k}) x_j \bar{x}_k \right) = 0,$$

for every $x \in \mathcal{S}_0$, which implies $c_{j,k} - c'_{j,k} = 0$ for all j, k . \square

2.7. Corollary. *Let $\mathcal{C} = (c_{j,k})$ (satisfying the above conditions). Then, the shift operator $S(x_j)_{j \in \mathbb{Z}} = (x_{j-1})_{j \in \mathbb{Z}}$ is an isometry on $\mathcal{B}_0^2(c_{j,k})$ if and only if $\mathcal{C} = (c_{j,k})$ is a Toeplitz matrix $c_{j,k} = c_{|j-k|}$, where (c_k) is a sequence satisfying $c_k \geq 0, \forall k \geq 1; c_0 = 0, 0 < \sum_k c_k < \infty$.*

Indeed,

$$\|Sx\|_{\mathcal{B}^2(\mathcal{C})}^2 = \sum_{j,k} c_{j,k} |x_{j-1} - x_{k-1}|^2 = \sum_{j,k} c_{j+1,k+1} |x_j - x_k|^2,$$

and if $\|Sx\|_{\mathcal{B}^2(\mathcal{C})}^2 = \|x\|_{\mathcal{B}^2(\mathcal{C})}^2$ for every $x \in \mathcal{S}_0$, we obtain by Lemma 2.6 $c_{j+1,k+1} = c_{j,k}$ for all j, k . Setting $c_j = c_{j,0}$ we get $c_{j,k} = c_{|j-k|}$.

Clearly, the converse is true as well. \square

3. LÉVY-KHINCHIN-SCHOENBERG WEIGHTS

The following lemma describes the spaces $L^2(\mathbb{T}, \mu)$ for which the norm

$$\|\mathcal{F}f\| := \|f\|_{L^2(\mu)}, \quad f \in \mathcal{P},$$

(here $\mathcal{P} = \operatorname{Lin}(z^n : n \in \mathbb{Z}), z \in \mathbb{T}$, is the set of all trigonometric polynomials) is a Besov-Dirichlet norm $\|\cdot\|_{\mathcal{B}^2(\mathcal{C})}$ on \mathcal{S}_0 (for a matrix \mathcal{C}). Note that $\mathcal{F}z^n = e_n$, and hence $\mathcal{F}\mathcal{P} = \mathcal{S}_0$. Speaking of the norms $\|\cdot\|_{\mathcal{B}^2(\mathcal{C})}$ we always suppose that the (Hermitian) matrix \mathcal{C} satisfies conditions of Section 2 (infinite \mathcal{C} -connected components E_k for all k , and $0 < \sum_j c_{j,k} < \infty$).

3.1. Lemma. *Let μ be a Borel measure on \mathbb{T} . The following statements are equivalent.*

(1) $x \mapsto \|\mathcal{F}^{-1}x\|_{L^2(\mu)}$ is a Besov-Dirichlet norm on \mathcal{S}_0 , $\mathcal{F}^{-1}x = \sum_k x_k e^{ikt}$.

(2) $\mu = w m$, where

$$w(e^{it}) = 4 \sum_{k \geq 1} c_k \sin^2(kt/2),$$

and $c_k \geq 0, 0 < \sum_{k=1}^{\infty} c_k < \infty$; in this case, $\mathcal{F}L^2(\mathbb{T}, w) = \mathcal{B}_0^2(c_{|j-k|}/2)$.

Proof. For (2) \Rightarrow (1), we simply observe that $4 \sin^2(kt/2) = |1 - e^{ikt}|^2$, and hence

$$\begin{aligned} \|\mathcal{F}^{-1}x\|_{L^2(w)}^2 &= \sum_{k \geq 1} c_k \int_{\mathbb{T}} |\mathcal{F}^{-1}x(e^{it})|^2 |1 - e^{ikt}|^2 dm \\ &= \sum_{k \geq 1} c_k \int_{\mathbb{T}} |\mathcal{F}^{-1}x(e^{it}) - e^{ikt} \mathcal{F}^{-1}x(e^{it})|^2 dm \\ &= \sum_{k \geq 1} c_k \int_{\mathbb{T}} |\mathcal{F}^{-1}x(e^{it}) - e^{ikt} \mathcal{F}^{-1}x(e^{it})|^2 dm = \sum_{k \geq 1} c_k \sum_{j \in \mathbb{Z}} |x_j - x_{j-k}|^2 \\ &= \frac{1}{2} \sum_{j, l \in \mathbb{Z}} c_{|j-l|} |x_j - x_l|^2 = \|x\|_{\mathcal{B}^2(\mathcal{C})}^2, \end{aligned}$$

where $c_{j,l} = c_{|j-l|}/2$.

For (1) \Rightarrow (2), we apply Corollary 2.7: the shift operator S is an isometry on $L^2(\mathbb{T}, \mu)$, and hence on $\mathcal{F}L^2(\mathbb{T}, \mu)$, and so if $x \mapsto \|\mathcal{F}^{-1}x\|_{L^2(\mu)}$ is a Besov-Dirichlet norm, $\|\mathcal{F}^{-1}x\|_{L^2(\mu)}^2 = \sum_{j, l \in \mathbb{Z}} c_{j,k} |x_j - x_k|^2$, the matrix $\mathcal{C} = (c_{j,k})$ is a Toeplitz one, that is, there exists a sequence (c_k) such that $c_{j,k} = c_{|j-k|}$. Now, the same computation as before but read in the opposite way shows that

$$\|\mathcal{F}^{-1}x\|_{L^2(\mu)}^2 = \sum_{j, l \in \mathbb{Z}} c_{|j-k|} |x_j - x_k|^2 = \|\mathcal{F}^{-1}x\|_{L^2(w)}^2,$$

for every polynomial $p = \mathcal{F}^{-1}x \in \mathcal{P}$, where $w(e^{it}) = \sum_{k \geq 1} c_k |1 - e^{ikt}|^2$. So

$$\int_{\mathbb{T}} |p|^2 d\mu = \int_{\mathbb{T}} |p|^2 w dm \quad (\forall p \in \mathcal{P}),$$

which obviously implies $\mu = wm$. \square

3.2. Comments. (1) Weights w of the type 3.1(2) first appeared in [Lev1934], [Khi1934] as characteristic exponents of stationary stochastic processes with independent increments and continuous time, nowadays often called Lévy processes. A vast theory and numerous applications of these processes are known, including deep connections with potential theory. The weights themselves were characterized by I. Schoenberg [Sch1938] (see also J. von Neumann and I. Schoenberg [vNS1941]):

- a non negative function $w \in C(\mathbb{T})$, $w(1) = 0$, is of the form $w(e^{it}) = 4 \sum_{k \geq 1} c_k \sin^2(kt/2)$, where $c_k \geq 0$, $\sum c_k < \infty$, if and only if w is “conditionally negative definite” in the following sense: $\sum_{j,k} w(z_j \bar{z}_k) a_j \bar{a}_k \leq 0$ for every choice of points $z_j \in \mathbb{T}$ and numbers $a_j \in \mathbb{C}$ such that $\sum_j a_j = 0$, or equivalently,
- if and only if $e^{-\epsilon w}$ is positive definite for every $\epsilon > 0$.

Schoenberg and von Neumann obtained these characterizations as a step in their solution of a metric geometry problem, in order to describe the so-called “screw lines” on a Hilbert space. The same class of functions appeared in the Beurling-Deny potential theory, see [BeD1958], [Den1970].

(2) Several properties of weights of this class (we call it the Lévy-Khinchin-Schoenberg class, LKS) are known; for instance $w \in \text{LKS} \Rightarrow w^\epsilon \in \text{LKS}$, and $1/(w^\epsilon)$ is positive definite if $0 < \epsilon \leq 1$ ([Sch1938], [vNS1941], [Kre1944]; see also [Lan1972], Sec. VI.3.13). It is also clear that

$$w(e^{it}) = \sum_{k \geq 1} c_k |1 - e^{ikt}|^2 = 0$$

if and only if e^{it} is a root of unity of order $d = \text{GCD}\{k: c_k > 0\}$, and so the zero set of w is always finite (if $w \not\equiv 0$). A generic $w \in \text{LKS}$ has only one zero at $e^{it} = 1$, but then $w(e^{idt})$ is again an LKS weight with zeros at the d -th roots of unity $e^{ikt/d}$, $k = 0, 1, \dots, d-1$. The infinite C -connected components property always holds for $\mathcal{FL}^2(\mathbb{T}, w)$, $w \in \text{LKS}$ (see Remark after Lemma 2.1).

We now derive first consequences of Lemma 3.1 and Section 2, in particular, a preliminary form of a description of the algebra $\text{Mult}(L^2(w))$ for $w \in \text{LKS}$. For this, we need the following simple lemma.

3.3. Lemma. *Let μ be a Borel measure on \mathbb{T} , and let $\mu = \mu_s + w\mu$ be its Lebesgue decomposition (μ_s is the singular part of μ , and $w \in L^1(\mathbb{T})$). The following statements are equivalent.*

(1) $\varphi_n : f \mapsto \hat{f}(n)$ (defined on trigonometric polynomials) extends to a bounded functional on $L^2(\mathbb{T}, \mu)$.

(2) $(z^k)_{k \in \mathbb{Z}}$ is a minimal sequence in $L^2(\mathbb{T}, \mu)$.

(3) $1/w \in L^1(\mathbb{T})$.

(4) $L^2(\mathbb{T}, w) \subset L^1(\mathbb{T})$.

Proof. It is clear that (1) \Leftrightarrow (2), (3) \Rightarrow (4) (by Cauchy’s inequality) and (4) \Rightarrow (1). Let us show that (1) \Rightarrow (3). Indeed, if $f \mapsto \hat{f}(0)$ is bounded, then there exists $g \in L^2(\mathbb{T}, \mu)$ such that $\int_{\mathbb{T}} f d\mu = \int_{\mathbb{T}} f g d\mu$ for every trigonometric polynomial f . Hence, $m = g\mu = g(\mu_s + w\mu)$, and so $g = 0$ μ_s -a.e. and $1 = gw$ m -a.e., which gives $\int_{\mathbb{T}} \frac{1}{w} dm = \int_{\mathbb{T}} g^2 w dm \leq \int_{\mathbb{T}} g^2 d\mu < \infty$. \square

3.4. Theorem. *Let $w(e^{it}) = 4 \sum_{k \geq 1} c_k \sin^2(kt/2)$ be an LKS weight, $c_k \geq 0$, $0 < \sum_k c_k < \infty$. Then $(\lambda_k) \in \text{Mult}(L^2(w))$ if and only if*

$$\lambda \in l^\infty(\mathbb{Z}) \quad \text{and} \quad \sum_k \left| \hat{f}(k) \right|^2 \mu_k^2 \leq C^2 \|f\|_{L^2(w)}^2, \quad \forall f \in \mathcal{P},$$

where $\mu_k^2 = \mu_k^2(\lambda) := \sum_j c_{|k-j|} |\lambda_j - \lambda_k|^2$, and the following alternative holds:

(1) either $\frac{1}{w} \in L^1(\mathbb{T})$, and then $\mathcal{S}_0 \subset \text{Mult}(L^2(w))$,

(2) or $\frac{1}{w} \notin L^1(\mathbb{T})$, and then $\text{Mult}(L^2(w))$ consists of all sequences constant on every C -connected component $E_k = k + D\mathbb{Z}$, $k = 0, \dots, D-1$, of \mathbb{Z} (see Remark after Lemma 2.1 for definitions); in particular, $\dim \text{Mult}(L^2(w)) = D < \infty$.

Proof. (1) Lemma 3.1 gives $\mathcal{F}L^2(\mathbb{T}, w) = \mathcal{B}_0^2(c_{|j-k|}/2)$, and by Lemma 3.3 φ_n are bounded on $\mathcal{B}_0^2(c_{|j-k|}/2)$, so that Lemmas 2.3 and 2.2 are applicable and yield the statement.

(2) The references to the same lemmas show that all functionals φ_n are unbounded, and consequently the space of multipliers is finitely dimensional (see Remark to Lemma 2.3 above for details). \square

Remark. The inequality

$$\sum_{k \in \mathbb{Z}} \left| \hat{f}(k) \right|^2 \mu_k^2 \leq C^2 \|f\|_{L^2(w)}^2, \quad \forall f \in \mathcal{P},$$

is equivalent to the embedding $\mathcal{F}L^2(\mathbb{T}, w) \subset l^2(\nu)$, where $\nu = (\mu_k^2)_{k \in \mathbb{Z}}$. Here and below we use the notation $l^2(\nu) = l^2(\mathbb{Z}, \nu)$ for the weighted l^2 space with norm

$$\|x\|_{l^2(\nu)} = \left(\sum_{k \in \mathbb{Z}} |x_k|^2 \nu_k \right)^{\frac{1}{2}},$$

where $\nu = (\nu_k)_{k \in \mathbb{Z}}$ ($\nu_k > 0$). For a capacity characterization of this embedding property see Section 4 below.

3.5. On the condition $1/w \in L^1(\mathbb{T})$ for LKS weights. The integrability condition $1/w \in L^1(\mathbb{T})$ plays a key role in the description of $\text{Mult}(L^2(\mathbb{T}, w))$ in Theorem 3.4. We discuss it below using the following simple observation.

3.6. Lemma. *Let $w(e^{it}) = 4 \sum_{k \geq 1} c_k \sin^2(kt/2)$ be an LKS weight, $c_k \geq 0$, $0 < \sum_k c_k < \infty$. Then,*

$$\frac{4t^2}{\pi^2} \sum_{k \leq \pi/t} c_k k^2 \leq w(e^{it}) \leq t^2 \sum_{k=1}^N c_k k^2 + 4 \sum_{k > N} c_k,$$

for $t \in (0, \pi)$ and for every $N \geq 0$. In particular,

$$c \sum_{n \geq 1} \frac{1}{\sum_{k=1}^n c_k k^2 + n^2 \sum_{k > n} c_k} \leq \int_{\mathbb{T}} \frac{dm}{w} \leq C \sum_{n \geq 1} \frac{1}{\sum_{k=1}^n c_k k^2},$$

with appropriate absolute constants $0 < c \leq C < \infty$.

Proof. For $0 \leq kt/2 \leq \pi/2$, one has $(kt/\pi)^2 \leq \sin^2(kt/2)$, and $\sin^2(kt/2) \leq (kt/2)^2$ for every k .

For the integral $\int_{-\pi}^{\pi} \frac{dt}{w(e^{it})}$, we first integrate around $t = 0$: $\int_{-\pi/N}^{\pi/N} = 2 \sum_{n \geq N} \int_{\pi/n+1}^{\pi/n}$ and

$$\begin{aligned} c \frac{\pi(1/n - 1/(n+1))}{(1/n^2) \sum_{k=1}^n c_k k^2 + \sum_{k > n} c_k} &\leq \int_{\pi/n+1}^{\pi/n} \frac{dm}{w} \\ &\leq C \frac{\pi(1/n - 1/(n+1))}{(1/n^2) \sum_{k=1}^n c_k k^2}, \end{aligned}$$

which gives the estimate claimed above if the only zero of $w(e^{it})$ is at $t = 0$. If there are other zeros of $w(e^{it})$ then, using comments 3.2(2), we can write $w(e^{it}) = w_1(e^{idt})$, where w_1 a LKS weight with the only zero at $t = 0$, and the inequalities follow from $\int_{\mathbb{T}} \frac{dm}{w} = \int_{\mathbb{T}} \frac{dm}{w_1}$. \square

3.7. Examples. As before, let $w(e^{it}) = 4 \sum_{k \geq 1} c_k \sin^2(kt/2)$.

(1) Let $c_1 = 1$, $c_k = 0$ for $k > 1$; then $w = 4 \sin^2(t/2)$. It follows that $1/w \notin L^1(\mathbb{T})$ and $\mathcal{FL}^2(\mathbb{T}, 4 \sin^2(t/2) dt) = \mathcal{B}_0^2(c_{|j-k|}/2)$, which is the completion of \mathcal{S}_0 in the norm

$$\|p\|_{L^2(w)} = \|\mathcal{F}p\|_{\mathcal{B}^2} = \left(\sum_{k \in \mathbb{Z}} |x_k - x_{k-1}|^2 \right)^{1/2},$$

for every polynomial $p \in \mathcal{P}$. Note that the completion $\mathcal{B}_0^2(c_{|j-k|}/2)$ is not a sequence space, but it can naturally be identified with $L^2(\mathbb{T}, 4 \sin^2(t/2) dt)$. The functionals φ_n are not continuous, and hence $\text{Mult}(L^2(w)) = \{\text{const}\}$. The same conclusion is still true for any finitely supported sequence $(c_k)_{k \geq 1}$, or for sequences “rapidly” tending to zero considered below.

(2) *Power-like kernels* $c_k \approx \frac{1}{k^{1+\alpha}}$ and the spaces $L^2(\mathbb{T}, |1 - e^{it}|^\alpha)$, $0 < \alpha < 2$. We use the notation $c_k \approx b_k$ in the following sense:

$$c_k \approx b_k \iff ab_k \leq c_k \leq Ab_k \text{ for large } k \text{ } (k \geq K) \text{ and } 0 < a \leq A < \infty.$$

Then, with appropriate constants $C > 0$ (which may be different in different entries) we have by 3.5 (for $0 < t < \pi$),

$$w(e^{it}) \leq Ct^2 \sum_{k \leq \pi/t} k^{1-\alpha} + C \sum_{k > \pi/t} 1/k^{1+\alpha} \leq Ct^\alpha + Ct^\alpha = Ct^\alpha,$$

$$w(e^{it}) \geq \frac{t^2}{\pi^2} \sum_{k \leq \pi/t} c_k k^2 \geq ct^2 \sum_{k \leq \pi/t} k^{1-\alpha} \geq ct^\alpha,$$

so that $w(e^{it}) \approx |t|^\alpha$ as $t \rightarrow 0$.

Conclusion: For $0 < \alpha < 1$, we have $1/w \in L^1(\mathbb{T})$, $\mathcal{FL}^2(\mathbb{T}, |1 - e^{it}|^\alpha) = \mathcal{B}_0^2(\frac{1}{|j-k|^{1+\alpha}})$ (with the equivalence of norms), $\mathcal{S}_0 \subset \text{Mult}(L^2(\mathbb{T}, |1 - e^{it}|^\alpha))$ and

$$(\lambda_k) \in \text{Mult}(L^2(\mathbb{T}, w)) \iff \lambda \in l^\infty(\mathbb{Z}) \text{ and } \mathcal{FL}^2(\mathbb{T}, |1 - e^{it}|^\alpha) \subset l^2(\nu),$$

where $\nu_k = \mu_k(\lambda)^2 := \sum_j \frac{|\lambda_j - \lambda_k|^2}{(|j-k|+1)^{1+\alpha}}$. (See Section 4 for a characterization of the last embedding.)

For $1 \leq \alpha < 2$, we have $1/w \notin L^1(\mathbb{T})$, and hence $\text{Mult}(L^2(w)) = \{\text{const}\}$. Clearly, for larger α ($\alpha \geq 2$) the preceding equality holds as well. \square

The following elementary lemma explains the condition $1/w \in L^1(\mathbb{T})$ for an LKS weight $w(e^{it}) = 4 \sum_{k \geq 1} c_k \sin^2(kt/2)$ in the “critical band” between $c_k = 1/k$ (decreasing too slowly since $\sum_k c_k = \infty$) and $c_k = 1/k^2$ (decreasing too fast since

$$\frac{2}{\pi^2} \sum_{k \geq 1} \frac{1}{k^2} \sin^2(kt/2) = \frac{t}{2\pi} (1 - \frac{t}{2\pi}), \quad 0 < t < 2\pi,$$

and so $1/w \notin L^1(\mathbb{T})$). By the way, the last observation shows that, for this integration question, without loss of generality we can assume that $\sum_k c_k k^2 = \infty$ (if not, then surely $1/w \in L^1(\mathbb{T})$).

3.8. Lemma. *Let $x \mapsto c(x)$ ($x \in [0, \infty)$) be a positive piecewise differentiable function such that $c_k = c(k)$, and*

$$x \mapsto x^\gamma c(x) \text{ eventually decreases for some } \gamma, 1 < \gamma < 3.$$

Then

$$w(e^{it}) \approx W(t) =: t^2 \int_0^{\pi/t} c(x) x^2 dx \quad \text{as } t \rightarrow 0,$$

and consequently

$$1/w \in L^1(\mathbb{T}) \Leftrightarrow \sum_{n \geq 1} \frac{1}{n \sum_{k=1}^n c_k k^2} < \infty.$$

Proof. First note that $\lim_{x \rightarrow \infty} xc(x) = 0$ and $\int_0^\infty c(x)dx < \infty$. By Lemma 3.6, it suffices to prove that there is a constant $C > 0$ such that

$$Ct^2 \sum_{k \leq \pi/t} c_k k^2 \geq \sum_{k > \pi/t} c_k, \quad \text{or} \quad Ct^2 \int_0^{1/t} c(x)x^2 dx \geq \int_{1/t}^\infty c(x)dx.$$

By the hypothesis $\gamma x^{\gamma-1}c(x) + x^\gamma c'(x) \leq 0$ (for $x \geq a > 0$), and hence $\gamma c(x) + xc'(x) \leq 0$. Integrating over $[y, b]$ and letting $b \rightarrow \infty$, we obtain

$$(\gamma - 1) \int_y^\infty c(x)dx - yc(y) \leq 0 \quad (\text{for } y \geq a).$$

Multiplying by y and integrating over $[a, s]$ we get

$$\begin{aligned} 0 &\geq (\gamma - 1) \int_a^s \left(\int_y^\infty c(x)dx \right) d(y^2/2) - \int_a^s y^2 c(y)dy \\ &= (\gamma - 1)2^{-1} \left(s^2 \int_s^\infty c(x)dx - a^2 \int_a^\infty c(x)dx + \int_a^s y^2 c(y)dy \right) - \int_a^s y^2 c(y)dy, \\ &\quad \frac{3-\gamma}{2} \int_a^s y^2 c(y)dy \geq (\gamma - 1)2^{-1} s^2 \int_s^\infty c(x)dx - \text{const}, \end{aligned}$$

which is equivalent to the inequality claimed above (with any constant $C > \frac{3-\gamma}{\gamma-1}$). \square

Remark. Yet another combination of “regularity conditions” on the behaviour of c_k as $k \rightarrow \infty$ leads to the following criterion:

Assume $c_k = c(|k|)$ ($k \in \mathbb{Z} \setminus \{0\}$) where $c : [1, \infty) \rightarrow \mathbb{R}_+$ is a function satisfying $c(t)t^2 \uparrow \infty$ (eventually) and $c(xy) \leq Ac(x)c(y)$ ($x, y \geq 1$) (in particular, all $c(t) = t^{-\gamma}$, $\gamma \leq 2$ satisfy these conditions). Then $\frac{1}{w} \in L^1(\mathbb{T})$ if and only if $\sum_{k \geq 1} \frac{1}{k^3 c_k} < \infty$ (or $\int_1^\infty \frac{dx}{x^3 c(x)} < \infty$).

Indeed, as in Lemma 3.8, we compare functions $B(x) = \int_0^x c(t)t^2 dt$ and $C(x) = x^2 \int_x^\infty c(t)dt$. We have

$$a \cdot c(x/2) \frac{x^3}{8} \leq \int_{x/2}^x c(t)t^2 dt \leq B(x) \leq A \cdot c(x)x^2 \int_1^x dt = Ac(x)x^3,$$

where $a > 0, A > 0$ are constants, and

$$C(x) = x^2 \int_x^\infty c(t)dt = x^3 \int_1^\infty c(xy)dy \leq a \cdot x^3 c(x) \int_1^\infty c(y)dy = A \cdot x^3 c(x).$$

So, by Lemma 3.6, if $\int_1^\infty \frac{dx}{x^3 c(x)} < \infty$, we obtain $\int_1^\infty \frac{dx}{B(x)} < \infty$, and hence $\frac{1}{w} \in L^1(\mathbb{T})$. Conversely, if $\frac{1}{w} \in L^1(\mathbb{T})$, then $\int_1^\infty \frac{dx}{B+C} < \infty$ and $\frac{1}{B+C} \geq \frac{1}{Ax^3 c(x) + Ax^3 c(x)}$, whence $\int_1^\infty \frac{dx}{x^3 c(x)} < \infty$. \square

3.9. On the Muckenhoupt condition $w \in (A_2)$ for LKS weights.

(1) Condition $w \in (A_2)$ is not so transparent as $1/w \in L^1(\mathbb{T})$ even for LKS weights. Recall that by definition

$$w \in (A_2) \Leftrightarrow \left(\frac{1}{|I|} \int_I w dm \right) \left(\frac{1}{|I|} \int_I \frac{1}{w} dm \right) \leq C, \quad \forall I \subset \mathbb{T},$$

where I is an arc (interval), and C is a constant which does not depend on I . Using P. Jones's (A_p) -factorization theorem ($w \in (A_p) \Leftrightarrow w = v_0 v_1^{1-p}$, where $v_0, v_1 \in (A_1)$), see [Duo2001], p.150), we get $w \in (A_2) \Leftrightarrow w = v_0/v_1$, where $v_0, v_1 \in (A_1)$. For a weight $w \in$ LKS, a good sufficient condition for $w \in (A_2)$ is simply $v = 1/w \in (A_1)$, which means that there exists $C > 0$ such that

$$Cv(x) \geq \frac{1}{|I|} \int_I v dm, \quad \text{for a.e. } x \in I,$$

for every arc (interval) $I \subset \mathbb{T}$.

Identifying $\mathbb{T} = (-\pi, \pi)$, it is easy to see that for $1/w \in (A_1)$ it suffices to check

$$\frac{C}{w(y)} \geq \frac{1}{y-x} \int_x^y \frac{dt}{w(t)} \quad \text{for all } 0 < x < y < \pi.$$

If the generating function $c(k) = c_k$ satisfies the condition of Lemma 3.8, one can replace w by W from this Lemma. The needed inequality $\frac{C}{W(y)} \geq \frac{1}{y-x} \int_x^y \frac{dt}{W(t)}$ (for all $0 < x < y < \pi$), follows from the following Hölder type condition (which defines “power-like” behaviour of W):

$$\frac{W(y)}{W(t)} \leq C(y/t)^\gamma, \quad \text{where } 0 < t < y < \pi \text{ and } 0 < \gamma < 1.$$

Indeed, the preceding condition implies

$$\begin{aligned} \frac{1}{y-x} \int_x^y \frac{W(y)}{W(t)} dt &\leq \frac{C}{y-x} \int_x^y (y/t)^\gamma dt \\ &= \frac{Cy^\gamma}{(y-x)(1-\gamma)} (y^{1-\gamma} - x^{1-\gamma}) = \frac{C}{1-\gamma} \frac{1 - (x/y)^{1-\gamma}}{1 - (x/y)} \leq \frac{C}{1-\gamma}. \end{aligned}$$

(2) LKS weights satisfying $1/w \in L^1(\mathbb{T})$ but not $w \in (A_2)$. In the notation of Lemma 3.8, let $c(x) = x^{-2}(\log(ex))^\beta$, $\beta > 1$. Then

$$w(e^{it}) \approx W(t) =: t^2 \int_0^{\pi/t} c(x)x^2 dx = t^2 \int_0^{\pi/t} (\log(ex))^\beta dx \approx t(\log(e/t))^\beta,$$

which gives $\int_0^1 \frac{dt}{w(e^{it})} < \infty$. On the other hand,

$$\int_0^1 \frac{dt}{w(e^{it})^{1+\epsilon}} \geq \text{const} \int_0^1 \frac{dt}{(t(\log(e/t))^\beta)^{1+\epsilon}} = \infty,$$

for every $\epsilon > 0$, and so $w \notin (A_2)$. \square

3.10. Typical asymptotic behaviour of LKS weights at 0. Consider an LKS weight $w(e^{it}) = 4 \sum_{k \geq 1} c_k \sin^2 \frac{kt}{2}$ as a function of the argument t , $-\pi \leq t \leq \pi$. The following claim shows that an arbitrary “mildly regular” (convexity-like) behaviour is permitted for an LKS weight as $t \rightarrow 0$. Since the coefficients $k \mapsto c_k$ are nearly monotone, the resulting function is always power-like: $|t|^2 \preceq w(e^{it}) \preceq |t|^\epsilon$ as $t \rightarrow 0$ for some $\epsilon > 0$. Here $\phi(t) \preceq \psi(t)$ means that $\phi(t) = O(\psi(t))$ as $t \rightarrow 0$.

Claim. *Let $u : (0, \infty) \rightarrow (0, \infty)$ be an (eventually) increasing piecewise differentiable function such that for some $-1 < \alpha < 1$ the function $s \mapsto s^\alpha u'(s)$ (eventually) decreases. Then there are $c_k > 0$, $\sum_k c_k < \infty$, such that*

$$w(e^{it}) \approx t^2 u(1/|t|) \text{ as } t \rightarrow 0.$$

In particular, $u(s) = s^\beta$, $0 < \beta < 2$, gives $w(e^{it}) \approx t^{2-\beta}$.

Indeed, define the function $c(\cdot)$ by $c(\pi y)(\pi y)^2 = u'(y)$, $y > 0$. It follows that, for $1 < \gamma =: \alpha + 2 < 3$, the function $c(\pi y)(\pi y)^\gamma = (\pi y)^{\gamma-2} u'(y)$ eventually decreases (and hence, $\int_0^\infty c(x) dx < \infty$). Then by Lemma 3.8 (for $t > 0$),

$$\begin{aligned} w(e^{it}) &\approx W(t) = t^2 \int_0^{\pi/t} c(x) x^2 dx = t^2 \int_0^{\pi/t} u'(x/\pi) dx = \\ &= \pi t^2 (u(1/t) - u(0)) \approx t^2 u(1/t) \text{ as } t \rightarrow 0. \end{aligned}$$

\square

3.11. Remarks on trivial multipliers for non-LKS weights. It is easy to see that $\text{Mult}(L^2(w)) = \{\text{const}\}$ implies $1/w \notin L^1(\mathbb{T})$ for every $w \in L^1(\mathbb{T})$ (not only for LKS weights). On the other hand, the converse is *generally not true for non-LKS weights* $w \in L^1(\mathbb{T})$: indeed, let

$$1/w = \sum_{k \in \mathbb{Z}} a_k |z - \alpha^k|^{-1},$$

where $\sum_k a_k < \infty$ ($a_k > 0$), $0 < \inf_k (a_k/a_{k+1}) \leq \sup_k (a_k/a_{k+1}) < \infty$, and let $\alpha \in \mathbb{T}$, $\alpha^k \neq 1$ ($\forall k \in \mathbb{Z}$). It is clear that the series converges a.e. (it is in $L^p(\mathbb{T})$ for every $0 < p < 1$) and $w \in L^\infty(\mathbb{T})$, but $1/w \notin L^1(\mathbb{T})$. Then it can be proved by the same reasoning as in [Nik2009] that the corresponding rotations

defined by $T_{\alpha^k} z^n = (\alpha^k)^n z^n$ ($\forall n \in \mathbb{Z}$) are (non-trivial) multipliers of $L^2(\mathbb{T}, w)$, and hence $\dim \text{Mult}(L^2(w)) = \infty$.

3.12. Remarks on duality of multipliers for general weights. Suppose $w^{\pm 1} \in L^1(\mathbb{T})$. Then

(1) $\lambda \in \text{Mult}(L^2(w)) \Leftrightarrow \bar{\lambda} \in \text{Mult}(L^2(1/w))$, where $\bar{\lambda} = (\bar{\lambda}_j)$.

Indeed, using the duality $\langle f, g \rangle = \int_{\mathbb{T}} f \bar{g} dm$ yields $(L^2(w))^* = L^2(1/w)$, and $T_{\lambda}^* = (\bar{\lambda}_j)_{j \in \mathbb{Z}}$.

(2) In general, $\lambda \in \text{Mult}(L^2(w)) \not\Leftrightarrow \bar{\lambda} \in \text{Mult}(L^2(w))$, and consequently

$$\text{Mult}(L^2(w)) \neq \text{Mult}(L^2(1/w)),$$

even for weights with the SLP (see Example 5.8 below). However, for LKS weights, obviously $\text{Mult}(L^2(w)) = \text{Mult}(L^2(1/w))$.

(3) For every weight $W \in L^1(\mathbb{T})$,

$$\bar{\lambda} \in \text{Mult}(L^2(W)) \Leftrightarrow \tilde{\lambda} \in \text{Mult}(L^2(W)) \Leftrightarrow \lambda \in \text{Mult}(L^2(\tilde{W})),$$

where $\tilde{\lambda} = (\lambda_{-j})$, and $\tilde{W}(z) = W(\bar{z})$ ($z \in \mathbb{T}$). In particular, if $w^{\pm 1} \in L^1(\mathbb{T})$, then

$$\text{Mult}(L^2(w)) = \text{Mult}(L^2(1/\tilde{w})).$$

Indeed,

$$\begin{aligned} \|T_{\bar{\lambda}} f\|_{L^2(W)}^2 &= \int_{\mathbb{T}} \left| \sum \bar{\lambda}_j \hat{f}(j) z^j \right|^2 W(z) dm = \int_{\mathbb{T}} \left| \sum \lambda_j \overline{\hat{f}(j)} z^{-j} \right|^2 W(z) dm \\ &= \int_{\mathbb{T}} \left| \sum \lambda_{-j} \overline{\hat{f}(-j)} z^j \right|^2 W(z) dm = \|T_{\tilde{\lambda}} \tilde{f}\|_{L^2(W)}^2, \end{aligned}$$

where f , and consequently $\tilde{f} = \sum \overline{\hat{f}(-j)} z^j$, is an arbitrary trigonometric polynomial, and $\|\tilde{f}\|_{L^2(W)} = \|\tilde{f}\|_{L^2(\tilde{W})} = \|f\|_{L^2(W)}$.

4. EMBEDDING OF BESOV-DIRICHLET SEQUENCE SPACES INTO WEIGHTED l^2 SPACES

In this section we continue to consider LKS weights w such that $1/w \in L^1(\mathbb{T})$, where

$$w(e^{it}) = 4 \sum_{k \geq 1} c_k \sin^2(kt/2),$$

and $c_k \geq 0$, $0 < \sum_{k=1}^{\infty} c_k < \infty$. To shorten the notation, we will denote by D the corresponding Besov-Dirichlet space:

$$D = \mathcal{B}_0^2(c_{|j-k|}/2) = \mathcal{F}L^2(\mathbb{T}, w).$$

Note that $w \in C(\mathbb{T})$, and consequently $D \supset l^2 = l^2(\mathbb{Z})$.

Here we present a characterization of the multiplier algebra $\text{Mult}(D) = \text{Mult}(L^2(w))$ for general LKS weights w such that $1/w \in L^1(\mathbb{T})$ in terms of *capacities* associated with D , as well as some *non-capacitary* characterizations for w satisfying additional “regularity” conditions (which includes standard power-like weights $|e^{it} - e^{i\theta}|^\alpha$, $0 < \alpha < 1$). To this end we will need some elements of potential theory, whose adaptation to our situation is presented below for the reader’s convenience.

As was shown in Sec. 3, $\lambda = (\lambda_k)_{k \in \mathbb{Z}} \in \text{Mult}(D)$ if and only if $\lambda \in l^\infty$, and the embedding $D \subset l^2(\nu)$ holds; the last condition is equivalent to the inequality

$$\|x\|_{l^2(\nu)} \leq C \|x\|_D, \quad \forall x \in \mathcal{S}_0,$$

where $\nu = \{\nu_j\}_{j \in \mathbb{Z}}$ is a nonnegative weight given by

$$\nu_j = \mu_j(\lambda)^2 = \sum_k c_{|j-k|} |\lambda_j - \lambda_k|^2.$$

In this section, we consider general embeddings of a given Besov-Dirichlet space $D \subset l^2(\nu) = l^2(\mathbb{Z}, \nu)$, where $\nu = (\nu_j)_{j \in \mathbb{Z}}$ is an arbitrary nonnegative weight on \mathbb{Z} , not necessarily related to a multiplier $\lambda \in \text{Mult}(D)$. Later on, we will be applying these results to multiplier related weights ν with $\nu_j = \mu_j(\lambda)^2$.

4.1. Green’s kernel. Let $w \in \text{LKS}$ and $1/w \in L^1(\mathbb{T})$. As was mentioned in Sec. 3, this yields that $w^\epsilon \in \text{LKS}$, and $1/(w^\epsilon)$ is positive definite, for all $0 < \epsilon \leq 1$. In particular, both $1/w$ and $1/(w^{1/2})$ are positive definite. Consider the discrete Green kernel $(g_{m-j})_{j, m \in \mathbb{Z}}$, where $g = \mathcal{F}(1/w) \geq 0$, so that

$$1/w(e^{it}) = \sum_{j \in \mathbb{Z}} g_j e^{ijt}.$$

The Green potential Gx is defined by $g * x$,

$$(Gx)_m = \sum_{j \in \mathbb{Z}} g_{m-j} x_j, \quad m \in \mathbb{Z}.$$

We will also need the corresponding potential operator Kx defined by $\kappa * x$,

$$(Kx)_m = \sum_{j \in \mathbb{Z}} \kappa_{m-j} x_j, \quad m \in \mathbb{Z},$$

where $\kappa = \mathcal{F}(1/w^{1/2}) \geq 0$, so that $\kappa * \kappa = g$.

It follows that both G and K have nonnegative symmetric kernels. Moreover, by Parseval’s theorem K is an isometry from l^2 onto D . Consequently, for any nonnegative weight $\nu = \{\nu_j\}_{j \in \mathbb{Z}}$, the embedding $D \subset l^2(\nu)$ is equivalent to the weighted norm inequality

$$\|Kx\|_{l^2(\nu)} \leq C \|x\|_{l^2}, \quad \forall x \in \mathcal{S}_0.$$

Since $G = K^2$ ($Gx = (\kappa * \kappa) * x = \kappa * (\kappa * x)$), the preceding inequality is equivalent to the corresponding weighted norm inequality for Green's potentials:

$$\|G(x\nu)\|_{l^2(\nu)} \leq C \|x\|_{l^2(\nu)}, \quad \forall x \in \mathcal{S}_0.$$

Here we use the notation $xy = (x_k y_k)_{k \in \mathbb{Z}}$.

4.2. Capacities and equilibrium potentials. The capacity of a nonempty set $J \subset \mathbb{Z}$ associated with the Besov-Dirichlet space D is given by ([FOT2011], Sec. 2.1):

$$\text{Cap}(J) := \inf \{ \|x\|_D^2 : x \in D, x_j \geq 1 \text{ if } j \in J \}.$$

In this definition we can restrict ourselves to x satisfying $0 \leq x \leq 1$, since by the contraction property $\|x\|_D \geq \|\bar{x}\|_D$, where $\bar{x} = \min[\max(x, 0), 1]$.

If the set of $x \in D$ such that $x \geq 1$ on J is empty then we set $\text{Cap}(J) = \infty$. This capacity can also be defined by means of the operator K with nonnegative kernel $\kappa = \mathcal{F}(1/w^{1/2})$ introduced above:

$$\text{Cap}(J) = \inf \{ \|y\|_{l^2}^2 : y \in l^2, y \geq 0, (Ky)_j \geq 1 \text{ if } j \in J \}.$$

A general theory of capacities associated with nonnegative kernels is presented in [AH1996], Sec. 2. Note that in our case the capacity of a single point set $J_0 = \{j_0\}$ is always positive:

$$\text{Cap}(J_0) = 1/\|1/w\|_{L^1(\mathbb{T})} > 0.$$

It follows from this and the contraction property that there exists a unique extremal element $x^J = Ky^J \in D$ such that $y^J \geq 0$, $0 \leq x^J \leq 1$, $x^J = 1$ on J , and

$$\|x^J\|_D^2 = \|y^J\|_{l^2}^2 = \text{Cap}(J),$$

provided $\text{Cap}(J) < \infty$. Moreover (see [AH1996], Sec. 2.5), at least for finite sets J , the capacity $\text{Cap}(\cdot)$ coincides with the dual Green capacity

$$\text{Cap}(J) = \sup \left\{ \sum_{j \in J} z_j : \text{supp } z \subset J, (Gz)_j \leq 1 \text{ if } j \in J, z \geq 0 \right\},$$

where the supremum is taken over all nonnegative sequences $z = (z_j)$ supported on J such that $Gz \leq 1$ on J . There exists a unique extremal element $z^J \geq 0$ supported on J such that $x^J = Ky^J = Gz^J \in D$, $Gz^J = 1$ on J , and $Gz^J \leq 1$ on \mathbb{Z} .

In summary (see [AH1996], [FOT2011]), to each finite nonempty set $J \subset \mathbb{Z}$ one can associate a unique extremal element (*equilibrium potential*) $x^J \in D$ such that

$$\begin{aligned} x^J &= Ky^J = Gz^J, \quad y^J = Kz^J, \\ x^J &= 1 \text{ on } J, \quad \|x^J\|_{l^\infty} = 1, \quad z^J \geq 0, \quad \text{supp } z^J \subset J, \\ \|x^J\|_D^2 &= \|y^J\|_{l^2}^2 = \|z^J\|_{l^1} = \text{Cap}(J). \end{aligned}$$

4.3. Theorem. Let $\nu = (\nu_k)_{k \in \mathbb{Z}}$ be a nonnegative sequence. Then the inequality

$$\sum_{j \in \mathbb{Z}} |x_k|^2 \nu_k \leq C \|x\|_D^2$$

holds for all $x \in D$ if and only if

$$\sum_{k \in J} \nu_k \leq C_1 \text{Cap}(J),$$

for every finite set $J \subset \mathbb{Z}$, where $C_1 \leq C \leq 4C_1$.

Theorem 4.3 is an immediate consequence of a discrete analogue of Maz'ya's strong capacity inequality stated in the following lemma. Its proof given below is based on an argument due to K. Hansson [Han1979] (see also [Maz2011], Sec. 11.2.2; [FOT2011], Sec. 2.4). Its main idea is a clever use of equilibrium potentials whose properties were discussed above.

4.4. Lemma. *Let $x = (x_j)_{j \in \mathbb{Z}} \in \mathcal{S}_0$. For $t > 0$, let $N_t = \{j \in \mathbb{Z} : |x_j| \geq t\}$. Then*

$$\int_0^\infty \text{Cap}(N_t) t \, dt \leq 2 \|x\|_D^2.$$

Proof. Clearly, the left-hand side of the preceding inequality is finite. Let $x = Ky$, where $y = \mathcal{F}((1/w)^{1/2} \mathcal{F}^{-1}x) \in l^2$. Notice that $|x| \leq K(|y|)$, and $y \in l^2$. Let $u^{N_t} = Gz^{N_t}$ be the equilibrium potential associated with the finite set N_t . Here $0 \leq u^{N_t} \leq 1$, $u^{N_t} = 1$ on N_t , and $z^{N_t} \geq 0$, $\text{supp } z^{N_t} \subset N_t$. Since $\text{Cap}(N_t) = \sum_j z_j^{N_t}$, and $(K|y|)_j \geq |x_j| \geq t$ on N_t , we have

$$\begin{aligned} \int_0^\infty \text{Cap}(N_t) t \, dt &= \int_0^\infty \sum_j z_j^{N_t} t \, dt \leq \int_0^\infty \sum_j (K|y|)_j z_j^{N_t} \, dt \\ &= \int_0^\infty \sum_j |y|_j (Kz^{N_t})_j \, dt = \sum_j |y|_j \int_0^\infty (Kz^{N_t})_j \, dt \\ &\leq \|y\|_{l^2} \left(\sum_j \left(\int_0^\infty (Kz^{N_t})_j \, dt \right)^2 \right)^{1/2}. \end{aligned}$$

We deduce

$$\begin{aligned} \sum_j \left(\int_0^\infty (Kz^{N_t})_j \, dt \right)^2 &= 2 \sum_j \int_0^\infty (Kz^{N_t})_j \int_0^t (Kz^{N_s})_j \, ds \, dt \\ &= 2 \int_0^\infty \int_0^t \sum_j (Kz^{N_t})_j (Kz^{N_s})_j \, ds \, dt. \end{aligned}$$

Since G and K have symmetric kernels, and $G = K^2$, we have

$$\sum_j (Kz^{N_t})_j (Kz^{N_s})_j = \sum_j (Gz^{N_s})_j z_j^{N_t}.$$

Here $Gz^{N_s} = u^{N_s}$ is the equilibrium potential associated with N_s . Consequently, $0 \leq (Gz^{N_s})_j \leq 1$ for all $j \in \mathbb{Z}$, and, since z^{N_t} is supported in N_t ,

$$\sum_j (Gz^{N_s})_j z_j^{N_t} \leq \sum_{j \in N_t} z_j^{N_t} = \text{Cap}(N_t).$$

Hence,

$$\sum_j \left(\int_0^\infty (Kz^{N_t})_j dt \right)^2 \leq 2 \int_0^\infty \int_0^t \text{Cap}(N_t) ds dt = 2 \int_0^\infty \text{Cap}(N_t) t dt.$$

Combining the preceding inequalities, we deduce

$$\left(\int_0^\infty \text{Cap}(N_t) t dt \right)^{1/2} \leq 2 \|y\|_{l^2} = 2 \|x\|_D.$$

□

Proof of theorem 4.3. To prove the “if” part of Theorem 4.3, we assume without loss of generality that $x \in \mathcal{S}_0$, and estimate

$$\begin{aligned} \sum_{j \in \mathbb{Z}} |x_k|^2 \nu_k &= 2 \int_0^\infty \left(\sum_{k: |x_k| \geq t} \nu_k \right) t dt \\ &\leq 2C_1 \int_0^\infty \text{Cap}(N_t) t dt \leq 4C_1 \|x\|_D^2. \end{aligned}$$

The “only if” part is obvious. □

4.6. A non-capacitary characterization of the embedding $D \subset l^2(\nu)$. The capacitary condition for the embedding $D \subset l^2(\nu)$ in Theorem 4.3 can be restated in the equivalent “energy” form which does not use capacities:

$$\sum_{j \in J} \sum_{m \in J} g_{m-j} \nu_j \nu_m \leq C \sum_{j \in J} \nu_j,$$

for every finite set $J \subset \mathbb{Z}$. (In the continuous case this was first noticed by D. R. Adams.) Indeed, denote by $\nu^J = \chi_J \nu$ the sequence ν restricted to J . Then, if the preceding condition holds, it follows that, for every $y \in l^2$ ($y \geq 0$) such that $Ky \geq 1$ on J ,

$$\begin{aligned} \sum_{j \in J} \nu_j &\leq \sum_{j \in J} (Ky)_j \nu_j = \sum_j y_j (K\nu^J)_j \leq \|y\|_{l^2} \|K\nu^J\|_{l^2} \\ &= \|y\|_{l^2} \left(\sum_{j \in J} \sum_{m \in J} g_{m-j} \nu_j \nu_m \right)^{1/2} \leq \|y\|_{l^2} C^{1/2} \left(\sum_{j \in J} \nu_j \right)^{1/2}. \end{aligned}$$

Consequently,

$$\sum_{j \in J} \nu_j \leq C \|y\|_{l^2}^2.$$

Minimizing over all such y , we obtain

$$\sum_{j \in J} \nu_j \leq C \operatorname{Cap}(J).$$

Conversely, suppose that the preceding condition holds. Obviously,

$$\sum_{j \in J} \sum_{m \in J} g_{m-j} \nu_j \nu_m = \|K\nu^J\|_{l^2}^2.$$

By duality,

$$\|K\nu^J\|_{l^2} = \sup_{y: \|y\|_{l^2} \leq 1} \left| \sum (K\nu^J)_j y_j \right|.$$

Since $\|y\|_{l^2} \leq 1$, invoking Theorem 4.3 we estimate

$$\left| \sum_j (K\nu^J)_j y_j \right| = \left| \sum_{j \in J} (Ky)_j \nu_j \right| \leq \|Ky\|_{l^2(\nu)} \left(\sum_{j \in J} \nu_j \right)^{1/2} \leq 2C^{1/2} \left(\sum_{j \in J} \nu_j \right)^{1/2}.$$

Thus,

$$\sum_{j \in J} \sum_{m \in J} g_{m-j} \nu_j \nu_m \leq 4C \sum_{j \in J} \nu_j.$$

□

4.7. Quasi-metric Green kernels. Suppose that the discrete Green's kernel (g_{j-m}) , where $g = \mathcal{F}(w^{-1})$ ($g = (g_j) > 0$) has the following quasi-metric property:

$$1/g_{j+m} \leq \varkappa (1/g_j + 1/g_m), \quad j, m \in \mathbb{Z},$$

for some constant $\varkappa > 0$. Then in the energy condition

$$\sum_{j \in J} \sum_{m \in J} g_{j-m} \nu_j \nu_m \leq C \sum_{j \in J} \nu_j,$$

which characterizes the embedding $D \subset l^2(\nu)$, it suffices to assume that J is a quasi-metric ball:

$$J = \{j \in \mathbb{Z} : g_{j-m} > 1/r\}, \quad m \in \mathbb{Z}, \quad r > 0.$$

Obviously, if (g_j) is nonincreasing for $j \geq 0$ then J is an interval: $\{j \in \mathbb{Z} : n_1 \leq j \leq n_2\}$.

This is a special case of a general result on quasi-metric kernels. Let (Ω, ν) be a measure space. A symmetric, measurable kernel $G : \Omega \times \Omega \rightarrow (0, +\infty]$ is called quasi-metric if $d = 1/G$ satisfies the quasi-triangle inequality

$$d(x, y) \leq \varkappa (d(x, z) + d(z, y))$$

for some $\varkappa > 0$ independent of $x, y, z \in \Omega$. By $B(x, r) = \{y \in \Omega : d(x, y) < r\}$ denote the quasi-metric ball of radius $r > 0$ centered at $x \in \Omega$. Consider the integral operator

$$Tf(x) = \int_{\Omega} G(x, y) f(y) d\nu(y), \quad x \in \Omega.$$

The following theorem is due to F. Nazarov.

4.8. Theorem. *Let (Ω, ν) be a measure space with σ -finite measure ν . Let G be a quasi-metric kernel on Ω , and let $d = 1/G$ be the corresponding quasi-metric such that $\nu(B) < \infty$ for every quasi-metric ball $B = B(x, r)$. Then*

$$\|Tf\|_{L^2(\Omega, \nu)} \leq C \|f\|_{L^2(\Omega, \nu)},$$

for all $f \in L^2(\Omega, \nu)$, if and only if there exists a constant $c = c(\kappa) > 0$ such that, for every B ,

$$\int_B \int_B G(x, y) d\nu(x) d\nu(y) \leq C_1 \nu(B).$$

Moreover, there exists a constant $c = c(\varkappa) > 0$ such that $C/c \leq C_1 \leq cC$.

In particular, this theorem is applicable to the weighted norm inequality for the discrete Green's operator with kernel (g_{j-m}) , where $g = \mathcal{F}(w^{-1})$:

$$\|G(x\nu)\|_{l^2(\nu)} \leq C \|x\|_{l^2(\nu)}, \quad \forall x \in \mathcal{S}_0,$$

provided $d = 1/g$ is a quasi-metric on \mathbb{Z} . As was demonstrated above, the preceding inequality is equivalent to the embedding $D \subset l^2(\nu)$.

The quasi-metric property holds in many important cases, in particular, for Green's kernel associated with the weight $w_{\alpha}(e^{it}) = |e^{it} - 1|^{\alpha}$ ($-1 < \alpha < 1$) discussed in the next subsection. For such weights, both capacity and non-capacity characterizations of multipliers $\text{Mult}(D) = \text{Mult}(L^2(w_{\alpha}))$ are available.

4.9. Example: Besov-Dirichlet spaces of fractional order. Let $f \in L^2(\mathbb{T}, w_{\alpha})$ where $w_{\alpha}(e^{it}) = |e^{it} - 1|^{\alpha}$ ($0 < \alpha < 1$). Then

$$w_{\alpha}(e^{it}) \approx 4 \sum_{j \geq 1} c_j \sin^2(jt/2), \quad \text{where } c_j = \frac{1}{(j+1)^{1+\alpha}}.$$

Let $x = \{x_j\}$, where $x_j = \hat{f}(j)$, $j \in \mathbb{Z}$. It follows that

$$\|f\|_{L^2(\mathbb{T}, w_{\alpha})}^2 \approx \|x\|_{D_{\alpha/2}}^2 = \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \frac{|x_j - x_m|^2}{(|j - m| + 1)^{1+\alpha}}.$$

Here $D_{\alpha/2} = D = \mathcal{B}_0^2(c_{|j-m|/2})$ is a Besov-Dirichlet space on \mathbb{Z} of fractional order α .

Next, a sequence $\lambda = (\lambda_j)_{j \in \mathbb{Z}}$ is a multiplier of $D_{\alpha/2}$:

$$\|\lambda x\|_{D_{\alpha/2}} \leq C \|x\|_{D_{\alpha/2}}, \quad \forall x \in D_{\alpha/2},$$

if and only $\lambda \in \text{Mult}(L^2(w_\alpha))$, or by duality $\lambda \in \text{Mult}(L^2(w_{-\alpha}))$.

One can rewrite this condition using the discrete Riesz potential $\mathcal{R}_{\alpha/2}$ of order $\alpha/2$, where

$$\mathcal{R}_{\alpha/2}y := \left(\sum_{j \in \mathbb{Z}} \frac{y_j}{(|m-j|+1)^{1-\alpha/2}} \right)_{m \in \mathbb{Z}},$$

in the following way (letting $x = \mathcal{R}_{\alpha/2}y$, $y \in \mathcal{S}_0$, and taking into account that $\mathcal{R}_{\alpha/2}\mathcal{S}_0$ is dense in $D_{\alpha/2}$):

$$\|\lambda(\mathcal{R}_{\alpha/2}y)\|_{D_{\alpha/2}} \leq C\|y\|_{l^2(\mathbb{Z})}, \quad \forall y \in \mathcal{S}_0.$$

The corresponding Green kernel $g = (g_{j-m})$ is equivalent to the Riesz kernel of order α since $g_j \approx 1/(|j|+1)^{1-\alpha}$, $j \in \mathbb{Z}$.

By Lemma 2.2 (1), $\lambda \in \text{Mult}(D_{\alpha/2})$ if and only if $\lambda \in l^\infty(\mathbb{Z})$, and the sequence $\mu = (\mu_j)_{j \in \mathbb{Z}}$ defined by:

$$\mu_j = \mu_j^\alpha(\lambda) := \left(\sum_{m \in \mathbb{Z}} \frac{|\lambda_j - \lambda_m|^2}{(|j-m|+1)^{1+\alpha}} \right)^{1/2}, \quad j \in \mathbb{Z},$$

is a multiplier from $D_{\alpha/2}$ to l^2 :

$$\sum_{j \in \mathbb{Z}} \mu_j^\alpha(\lambda)^2 |x_j|^2 \leq C^2 \|x\|_{D_{\alpha/2}}^2, \quad \forall x \in D_{\alpha/2}.$$

Equivalently, $D_{\alpha/2} \subset l^2(\nu)$ where $\nu = \mu^2$. The preceding inequality holds if and only if μ obeys the capacity condition of Theorem 4.3:

$$\sum_{j \in J} \mu_j^\alpha(\lambda)^2 \leq C \text{Cap}_\alpha(J),$$

for every finite set $J \subset \mathbb{Z}$. Here $\text{Cap}_\alpha(\cdot)$ is the capacity associated with the Besov-Dirichlet space $D = D_{\alpha/2}$ (see Sec. 4.2).

Since the corresponding Green kernel (g_{j-m}) has the quasi-metric property, and g_j is decreasing for $j \geq 0$, it follows from Theorem 4.8 that the embedding $D_{\alpha/2} \subset l^2(\nu)$ ($\nu = \mu^2$) is equivalent to the energy condition

$$\sum_{j \in J} \sum_{m \in J} \frac{\mu_j^\alpha(\lambda)^2 \mu_m^\alpha(\lambda)^2}{(|j-m|+1)^{1-\alpha}} \leq C \sum_{j \in J} \mu_j^\alpha(\lambda)^2,$$

for every *interval* J in \mathbb{Z} . This characterization of multipliers $\mu: D_{\alpha/2} \rightarrow l^2$ is due to Kalton and Tzafriri [KT1998].

4.10. Multipliers in pairs of Besov-Dirichlet spaces. To treat weights with several power-like singularities considered below, we will need classes of multipliers acting from $D_{\beta/2}$ to $D_{\alpha/2}$. They will be denoted by $\text{Mult}(D_{\beta/2} \rightarrow D_{\alpha/2})$; for $\alpha = \beta$ we will continue to use the notation $\text{Mult}(D_{\alpha/2})$. We remark that $\text{Mult}(D_{\beta/2} \rightarrow D_{\alpha/2})$ coincides with the class of Fourier multipliers

$\text{Mult}(L^2(w_\beta) \rightarrow L^2(w_\alpha))$ defined in a similar way. The following characterization of multipliers is similar to the continuous case (see [MSh2009]), but there are certain differences which we need to take into account (see Remark 4.12 below).

4.11. Theorem. (1) *Let $0 < \beta \leq \alpha < 1$. Then $\lambda \in \text{Mult}(D_{\beta/2} \rightarrow D_{\alpha/2})$ if and only if $\lambda \in l^\infty$, and $\mu \in \text{Mult}(D_{\beta/2} \rightarrow l^2)$, where $\mu = (\mu_j)$ is defined by*

$$\mu_j = \mu_j^\alpha(\lambda) := \left(\sum_m \frac{|\lambda_j - \lambda_m|^2}{(|j - m| + 1)^{1+\alpha}} \right)^{1/2}, \quad j \in \mathbb{Z}.$$

Equivalently,

$$\lambda \in l^\infty, \quad \text{and} \quad \sum_{j \in J} \mu_j^\alpha(\lambda)^2 \leq C \text{Cap}_\beta(J),$$

for every finite set $J \subset \mathbb{Z}$, where C does not depend on J .

(2) *Let $0 < \alpha < \beta < 1$. Then $\lambda \in \text{Mult}(D_{\beta/2} \rightarrow D_{\alpha/2})$ if and only if $\lambda \in \text{Mult}(D_{(\beta-\alpha)/2} \rightarrow l^2)$, and $\mu \in \text{Mult}(D_{\beta/2} \rightarrow l^2)$. Equivalently,*

$$\sum_{j \in J} |\lambda_j|^2 \leq C \text{Cap}_{\beta-\alpha}(J), \quad \text{and} \quad \sum_{j \in J} \mu_j^\alpha(\lambda)^2 \leq C \text{Cap}_\beta(J),$$

for every finite set $J \subset \mathbb{Z}$, where C does not depend on J .

Proof. If $\lambda \in \text{Mult}(L^2(w_\beta) \rightarrow L^2(w_\alpha))$, and $\|T_\lambda\| = \|T_\lambda\|_{L^2(w_\beta) \rightarrow L^2(w_\alpha)}$ is the multiplier norm, then for all $n \in \mathbb{Z}$,

$$|\lambda_n| \cdot \|w_\alpha\|_{L^1(\mathbb{T})}^{1/2} = \|T_\lambda z^n\|_{L^2(w_\alpha)} \leq \|T_\lambda\| \|z^n\|_{L^2(w_\beta)} = \|T_\lambda\| \cdot \|w_\beta\|_{L^1(\mathbb{T})}^{1/2}.$$

Consequently, $\lambda \in l^\infty$.

(1) Suppose $0 < \beta \leq \alpha < 1$, and $\lambda \in \text{Mult}(L^2(w_\beta) \rightarrow L^2(w_\alpha))$. Since $\lambda \in l^\infty$, we have

$$\sum_{j \in \mathbb{Z}} |\lambda_j|^2 \mu_j^\alpha(x)^2 = \sum_{j \in \mathbb{Z}} |\lambda_j|^2 \sum_{m \in \mathbb{Z}} \frac{|x_j - x_m|^2}{(|j - m| + 1)^{1+\alpha}} \leq C \|x\|_{D_{\alpha/2}}^2 \leq C \|x\|_{D_{\beta/2}}^2.$$

Hence, as in the case $\alpha = \beta$ considered above, we see that $\lambda \in \text{Mult}(L^2(w_\beta) \rightarrow L^2(w_\alpha))$ if and only if $\lambda \in l^\infty$, and

$$\sum_{j \in \mathbb{Z}} |x_j|^2 \mu_j^\alpha(\lambda)^2 = \sum_{j \in \mathbb{Z}} |x_j|^2 \sum_{m \in \mathbb{Z}} \frac{|\lambda_j - \lambda_m|^2}{(|j - m| + 1)^{1+\alpha}} \leq C \|x\|_{D_{\beta/2}}^2,$$

for all $x \in D_{\beta/2}$, or equivalently $\mu \in \text{Mult}(D_{\beta/2} \rightarrow l^2)$. By Theorem 4.3 the preceding condition is equivalent to

$$\sum_{j \in J} \mu_j^\alpha(\lambda)^2 \leq C \text{Cap}_\beta(J),$$

for every finite set $J \subset \mathbb{Z}$.

(2) Suppose $0 < \alpha < \beta < 1$, and $\lambda \in \text{Mult}(L^2(w_\beta) \rightarrow L^2(w_\alpha))$. Then by duality, $\lambda \in \text{Mult}(L^2(w_{-\alpha}) \rightarrow L^2(w_{-\beta}))$. Hence, using interpolation for operators acting in L^2 spaces with weights, we obtain $\lambda \in \text{Mult}(L^2(w_{\beta-\alpha}) \rightarrow L^2)$. By Theorem 4.3 this implies

$$\sum_{j \in J} |\lambda_j|^2 \leq C \text{Cap}_{\beta-\alpha}(J),$$

for all $J \subset \mathbb{Z}$. On the other hand, as in the case $\alpha \geq \beta$, we have that $\lambda \in \text{Mult}(L^2(w_\beta) \rightarrow L^2(w_\alpha))$ if, for all $x \in D_{\beta/2}$, the following pair of inequalities hold:

$$\sum_{j \in \mathbb{Z}} |x_j|^2 \mu_j^\alpha(\lambda)^2 \leq C \|x\|_{D_{\beta/2}}^2,$$

$$\sum_{j \in \mathbb{Z}} |\lambda_j|^2 \mu_j^\alpha(x)^2 \leq C \|x\|_{D_{\beta/2}}^2.$$

Moreover, if the second inequality holds, then the first one is necessary in order that $\lambda \in \text{Mult}(L^2(w_\beta) \rightarrow L^2(w_\alpha))$.

It remains to show that if $\lambda \in \text{Mult}(L^2(w_{\beta-\alpha}) \rightarrow L^2)$, then the last inequality holds. Clearly, if $\lambda \in \text{Mult}(L^2(w_{\beta-\alpha}) \rightarrow L^2)$ is a bounded Fourier multiplier, or equivalently, $\lambda \in \text{Mult}(D_{(\beta-\alpha)/2} \rightarrow l^2)$, it follows that

$$\|\lambda \mu^\alpha(x)\|_{l^2} \leq C \|\mu^\alpha(x)\|_{D_{(\beta-\alpha)/2}},$$

where $\mu^\alpha(x) = (\mu_j^\alpha(x))$. We note that by definition

$$\|\mu^\alpha(x)\|_{D_{(\beta-\alpha)/2}} = \|\mu^{\beta-\alpha}[\mu^\alpha(x)]\|_{l^2}.$$

Let us show that

$$\|\mu^{\beta-\alpha}[\mu^\alpha(x)]\|_{l^2} \leq C \|\mu^\beta(x)\|_{l^2},$$

where C depends only on α, β . This is a discrete analogue of Lemma 4.2.1 [MSh2009]: by the triangle inequality,

$$\|\mu^{\beta-\alpha}[\mu^\alpha(x)]\|_{l^2}^2 \leq \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \frac{|x_{j+k} - x_k + x_{m+k} - x_{j+k+m}|^2}{(|j|+1)^{1+\alpha}(|m|+1)^{1+\beta-\alpha}}.$$

The triple sum on the right-hand side is symmetric with respect to j and m , and so it is enough to consider the case $|m| \geq |j|$. We estimate

$$\sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \frac{|x_{j+k} - x_k|^2}{(|j|+1)^{1+\alpha}} \sum_{|m| \geq |j|} \frac{1}{(|m|+1)^{1+\beta-\alpha}} \leq C \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \frac{|x_{j+k} - x_k|^2}{(|j|+1)^{1+\beta}} = C \|\mu^\beta(x)\|_{l^2}^2.$$

Similarly, interchanging the order of summation in the remaining term and replacing k with $n = k + m$, we estimate

$$\sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \frac{1}{(|j|+1)^{1+\alpha}} \sum_{|m| \geq |j|} \frac{|x_{m+k} - x_{j+k+m}|^2}{(|m|+1)^{1+\beta-\alpha}} \leq C \|\mu^\beta(x)\|_{l^2}^2.$$

Combining the above estimates and taking into account that $\|\mu^\beta(x)\|_{l^2} = \|x\|_{D_{\beta/2}}^2$, we conclude that $\lambda \in \text{Mult}(D_{(\beta-\alpha)/2} \rightarrow L^2)$ implies

$$\|\lambda \mu^\alpha(x)\|_{l^2} \leq C \|x\|_{D_{\beta/2}}^2,$$

for all $x \in D_{\beta/2}$. Thus, $\lambda \in \text{Mult}(D_{\beta/2} \rightarrow D_{\alpha/2})$ if and only if $\mu = \mu^\alpha(\lambda) \in \text{Mult}(D_{\beta/2} \rightarrow l^2)$ and $\lambda \in \text{Mult}(D_{(\beta-\alpha)/2} \rightarrow l^2)$. By Theorem 4.3 both conditions have the corresponding capacity characterizations (see Sec. 4.9). \square

4.12. Remark. In the continuous case of multipliers in pairs of Sobolev spaces $W^{\alpha,p}(\mathbb{R}^n)$, it is known that $\text{Mult}(W^{\beta,p}(\mathbb{R}^n) \rightarrow W^{\alpha,p}(\mathbb{R}^n)) = \{0\}$ if $\alpha > \beta > 0$ (see [MSh2009], Sec. 2.1), contrary to the discrete case $\text{Mult}(D_{\beta/2} \rightarrow D_{\alpha/2})$.

5. WEIGHTS WITH SEVERAL LKS SINGULARITIES

In this section we are concerned with weights which are equivalent to products, or sums of the reciprocals of LKS weights. Such weights may have finitely many singularities, and generally are no longer LKS-weights. Nevertheless, we will be able to characterize multipliers $\text{Mult}(L^2(w))$, and show that they obey the Spectrum Localization Property (SLP). It is known that the SLP fails for weights with infinitely many singularities of this type (see 5.12-5.13 below).

In this section it will be convenient to use the following notation for weights on \mathbb{T} : $w_\alpha = |e^{it} - 1|^\alpha$, and $w_\alpha^\theta = |e^{it} - e^{i\theta}|^\alpha$ ($\alpha \in \mathbb{R}$). We will consider weights of the type

$$w = \prod_{j=0}^{d-1} w_{\alpha_j}^{\theta_j},$$

where $\theta_j \in \mathbb{R}$ are pairwise distinct $(\text{mod}(2\pi))$ points.

5.1. Weights with several singularities of the same order.

Theorem. Let $w = \prod_{j=0}^{d-1} w_{\alpha_j}^{\theta_j}$, where θ_j are pairwise distinct $(\text{mod}(2\pi))$ points.

Let $d = d_1 > d_2 > \dots > d_n = 1$ be all distinct divisors of d . We denote by d_s the largest among the divisors such that the set of singularities $\sigma = \{e^{i\theta_j} : 0 \leq j < d-1\}$ is a union of vertices of $n_s = d/d_s$ distinct regular d_s -sided polygons. Then $\text{Mult}(L^2(w)) = \text{Mult}(L^2(w_\alpha(e^{itd_s})))$, and $\lambda \in \text{Mult}(L^2(w))$ if and only if $\lambda \in l^\infty$, and

$$\sum_{j \in J} \sum_{\substack{m \in \mathbb{Z} \\ d_s | j-m}} \frac{|\lambda_j - \lambda_m|^2}{(|j-m|+1)^{1+\alpha}} \leq C \text{Cap}_\alpha(J),$$

for every finite set $J \subset \mathbb{Z}$.

Remarks. (1) If $d_s = 1$, i.e., σ has no rotational symmetries (the generic case), then $\text{Mult}(L^2(w)) = \text{Mult}(L^2(w_\alpha))$.

(2) If d is a prime number then either $d_s = d$ or $d = 1$. However, generally d_s is not necessarily a prime number. For instance, if $d = 12$, then σ may consist of the vertices of either a single regular dodecagon, or two regular hexagons, or three squares, or four regular triangles, or six 2-gons, i.e., pairs of opposite points. Then $d_s = 12, 6, 4, 3, 2$, respectively, and there are $n_s = d/d_s$ distinct regular polygons, so that each of them is a rotation of the set of roots of unity of order d_s . If $d_s = 1$, then σ consists of 12 points on \mathbb{T} with no rotational symmetry.

Proof. 1. For the sake of simplicity, we first consider the case of two singularities. Let $w = w_{-\alpha}w_{-\alpha}^\theta$, where $0 < \alpha < 1$ and $\theta \neq 0 \pmod{2\pi}$. Note that

$$w \approx w_{-\alpha} + w_{-\alpha}^\theta = |e^{it} - 1|^{-\alpha} + |e^{it} - e^{i\theta}|^{-\alpha}.$$

By duality $\text{Mult}(L^2(w)) = \text{Mult}(L^2(1/\tilde{w}))$, where $\tilde{w}(e^{it}) = w(e^{-it})$. Hence, at the same time we obtain a characterization of multipliers for weights of the type

$$1/w = w_\alpha w_\alpha^\theta = |e^{it} - 1|^\alpha |e^{it} - e^{i\theta}|^\alpha.$$

It will be more convenient to work with convolution operators $T_\lambda f = k \star f$ on $L^2(w)$, where $\lambda = (\lambda_j)_{j \in \mathbb{Z}} = \mathcal{F}k \in l^\infty(\mathbb{Z})$, and consequently k is a pseudo-measure on \mathbb{T} . For a pseudo-measure k on \mathbb{T} , it follows that $\lambda = \mathcal{F}k \in \text{Mult}(L^2(w))$ if and only if

$$\|k \star f\|_{L^2(w)} \leq C \|f\|_{L^2(w)},$$

for all trigonometric polynomials f , which is equivalent to a pair of inequalities:

$$\|k \star f\|_{L^2(w_{-\alpha})} \leq C \|f\|_{L^2(w_{-\alpha}w_{-\alpha}^\theta)},$$

$$\|k \star f\|_{L^2(w_{-\alpha}^\theta)} \leq C \|f\|_{L^2(w_{-\alpha}w_{-\alpha}^\theta)}.$$

Using the rotation operator $R_\theta f(e^{it}) = f(e^{i(t+\theta)})$, and letting $g = R_\theta f$, we see that the second inequality is equivalent to

$$\|k \star g\|_{L^2(w_{-\alpha})} \leq C \|g\|_{L^2(w_{-\alpha}w_{-\alpha}^{-\theta})},$$

for all $g \in L^2(w_{-\alpha}w_{-\alpha}^{-\theta})$.

By duality, we deduce that $\lambda \in \text{Mult}(L^2(w))$ if and only if the following pair of inequalities hold:

$$\|\tilde{k} \star f\|_{L^2(w_\alpha w_\alpha^\theta)} \leq C \|f\|_{L^2(w_\alpha)},$$

$$\|\tilde{k} \star f\|_{L^2(w_\alpha w_\alpha^{-\theta})} \leq C \|f\|_{L^2(w_\alpha)},$$

for all $f \in L^2(w_\alpha)$, where $\tilde{k}(e^{ix}) = k(e^{-ix})$.

Let $W = w_\alpha(w_\alpha^\theta + w_\alpha^{-\theta})$. Adding up the preceding displayed inequalities, we obtain that they are equivalent to:

$$\|\tilde{k} \star f\|_{L^2(W)} \leq C \|f\|_{L^2(w_\alpha)}, \quad \forall f \in L^2(w_\alpha).$$

Clearly, if $\theta \neq \pi + 2\pi n$ ($n \in \mathbb{Z}$), i.e. in the generic case, we have

$$W(e^{it}) = |e^{it} - 1|^\alpha (|e^{it} - e^{i\theta}|^\alpha + |e^{it} - e^{-i\theta}|^\alpha) \approx |e^{it} - 1|^\alpha = w_\alpha.$$

Hence in this case $T_\lambda = k \star (\cdot)$ is a bounded operator in $L^2(w)$ if and only if $\tilde{k} \star (\cdot)$ is a bounded operator in $L^2(w_\alpha)$, or equivalently $\lambda \in \text{Mult}(L^2(w_\alpha))$, since $\tilde{w}_\alpha = w_\alpha$. Thus, multipliers for weights with two generic singularities are the same as for weights with one singularity (characterized in Section 4).

In the non-generic case $\theta = \pi + 2\pi n$ ($n \in \mathbb{Z}$), we have

$$W(e^{it}) \approx |e^{it} - 1|^\alpha |e^{it} + 1|^\alpha = |e^{2it} - 1|^\alpha,$$

which is equivalent to an LKS weight. It follows that $\lambda \in \text{Mult}(L^2(w))$ if and only if $\lambda \in l^\infty(\mathbb{Z})$, and

$$\sum_{\substack{j \in \mathbb{Z} \\ m \in \mathbb{Z} \\ j-m \text{ even}}} \frac{|\lambda_j x_j - \lambda_m x_m|^2}{(|j-m|+1)^{1+\alpha}} \leq C \|x\|_{D_{\alpha/2}}^2, \quad \forall x \in D_{\alpha/2}.$$

Since $\lambda \in l^\infty(\mathbb{Z})$, it is easy to see, using the same argument as in the case of the weight $|e^{it} - 1|^\alpha$, that the preceding inequality holds if and only if

$$\sum_{\substack{j \in \mathbb{Z} \\ m \in \mathbb{Z} \\ j-m \text{ even}}} |x_j|^2 \frac{|\lambda_j - \lambda_m|^2}{(|j-m|+1)^{1+\alpha}} \leq C \|x\|_{D_{\alpha/2}}^2, \quad \forall x \in D_{\alpha/2}.$$

Letting

$$\mu_j^\alpha(\lambda)^2 = \sum_{\substack{m \in \mathbb{Z} \\ j-m \text{ even}}} \frac{|\lambda_j - \lambda_m|^2}{(|j-m|+1)^{1+\alpha}}, \quad j \in \mathbb{Z},$$

we see that the multiplier problem is reduced to the inequality

$$\sum_{j \in \mathbb{Z}} \mu_j^\alpha(\lambda)^2 |x_j|^2 \leq C \|x\|_{D_{\alpha/2}}^2, \quad \forall x \in D_{\alpha/2}.$$

Inequalities of this type have been characterized in terms of Besov-Dirichlet capacities, or energies associated with $D_{\alpha/2}$ (Sec. 4.9).

2. A similar argument works for any number of singularities d . Let $J_d = \{0, 1, \dots, d-1\}$, and let $\sigma = \{e^{i\theta_j}\}_{j \in J_d}$ be the set of singularities of the weight $w = \prod_{j \in J_d} w_{-\alpha}^{\theta_j}$, where $\theta_j \in [0, 2\pi)$ are pairwise distinct points, and $0 < \alpha < 1$. Note that $w \approx \sum_{j \in J_d} w_{-\alpha}^{\theta_j}$. (Analogous results for weights of the type $w = \prod_{j \in J_d} w_\alpha^{\theta_j}$ follow by duality.)

Using the same argument as in the case $d = 2$, it is easy to see that $\lambda \in \text{Mult}(L^2(w))$ if and only if the following inequality holds:

$$\|\tilde{k} \star f\|_{L^2(W)} \leq C \|f\|_{L^2(w_\alpha)}, \quad \forall f \in L^2(w),$$

where

$$W = \sum_{j \in J_d} \prod_{m \in J_d} w_\alpha^{\theta_m - \theta_j}.$$

To complete the proof of the Theorem we will need the following lemma which describes the set of singularities of W on \mathbb{T} . As we will see, this question is related to actions of the group of rotations on the finite set $\sigma \subset \mathbb{T}$.

Lemma. *Let $W = \sum_{j \in J_d} \prod_{m \in J_d} w_\alpha^{\theta_m - \theta_j}$ be the weight defined above. The following alternative holds.*

- (i) *Either there exists a divisor D of d , $1 < D \leq d$, such that $\sigma = \{e^{i\theta_j}\}_{j \in J_d}$ is the union of d/D (different) regular D -sided polygons, and then, denoting by d_s the maximal possible such D , we have $W \approx |1 - e^{id_s t}|^\alpha$,*
- (ii) *or $W \approx |1 - e^{it}|^\alpha$.*

Remark. We wish to thank Stephen Montgomery-Smith for pointing out that this Lemma and its proof given below are related to the orbit-stabilizer theorem and Burnside lemma (see [Ja1985], Sec. 1.12), and can be generalized to arbitrary abelian groups.

Let us complete the proof of the Theorem assuming the Lemma. In case (i) of the Lemma, σ is the union of $n_s = d/d_s$ (different) regular d_s -sided polygons, and W is equivalent to the LKS weight $w_\alpha(e^{i\theta d_s}) = |e^{i\theta d_s} - 1|^\alpha$. Letting

$$\mu_j^\alpha(\lambda)^2 = \sum_{\substack{m \in \mathbb{Z} \\ d_s \mid j-m}} \frac{|\lambda_j - \lambda_m|^2}{(|j-m|+1)^{1+\alpha}}, \quad j \in \mathbb{Z},$$

and using the same argument as above we see that in this case $\lambda \in \text{Mult}(L^2(w))$ if and only if $\lambda \in l^\infty(\mathbb{Z})$, and

$$\sum_{j \in \mathbb{Z}} \mu_j^\alpha(\lambda)^2 |x_j|^2 \leq C \|x\|_{D_{\alpha/2}}^2, \quad \forall x \in D_{\alpha/2}.$$

In case (ii), σ has no rotational symmetry, and $W \approx w_\alpha$, so that $\text{Mult}(L^2(w)) = \text{Mult}(L^2(w_\alpha))$, and the preceding characterization holds with $d_s = 1$. \square

Proof of the Lemma. 1. Without loss of generality we may assume that $\theta_0 = 0$. Clearly, $t = 0$ is a zero of $W(e^{it})$. Suppose there exists $t \in (0, 2\pi)$ such that $W(e^{it}) = 0$. Then for every $j \in J_d$ there exists $m \in J_d$ such that $w_\alpha^{\theta_m - \theta_j}(e^{it}) = 0$. In other words, there is a permutation $j \mapsto m(j)$ of J_d such that, for all $j \in J_d$, we have $\theta_{m(j)} = \theta_j + t \pmod{2\pi}$, where $m(j)$ is unique, and $m(j_1) \neq m(j_2)$ if $j_1 \neq j_2$. Obviously, $m(j) \neq j$ since $t \neq 0$. Adding together

these equations for all $j \in J_d$, we see that $td = 0 \pmod{2\pi}$, i.e., $t = 2\pi n/d$ for some $n \in J_d$. It follows that

$$\theta_{m(j)} = \theta_j + \frac{2\pi n}{d} \pmod{2\pi}, \quad \forall j \in J_d.$$

Moreover, for the consecutive iterations of the map $j \mapsto m(j)$ defined by $m^{(0)}(j) = j$, and $m^{(k)}(j) = m(m^{(k-1)}(j))$ ($k = 1, 2, \dots$), we see that all $e^{i\theta_{m^{(k)}(j)}} \in \sigma$, where

$$\theta_{m^{(k)}(j)} = \theta_j + \frac{2\pi kn}{d} \pmod{2\pi}, \quad \forall j \in J_d, \quad k = 0, 1, 2, \dots$$

2. Suppose first that d is a prime number. Then $z_k = m^{(k)}(0) = e^{2\pi i kn/d} \in \sigma$ ($k = 0, 1, 2, \dots$) are obviously roots of unity of order d . It is easy to see that z_k are distinct for $k \in J_d$. Indeed, if $k_1, k_2 \in J_d$, and $z_{k_1} = z_{k_2}$ for $k_2 > k_1$, we have

$$2\pi(k_2 - k_1)\frac{n}{d} = 0 \pmod{2\pi}.$$

Since d is a prime number, and $0 < k_2 - k_1 \leq d - 1$, it follows that $n = 0$ and consequently $t = 0$, which is a contradiction. Hence $\sigma = \{z_k\}_{k \in J_d}$ consists of all the roots of unity of order d . In this case obviously $W(z) = 0$ if and only if z is a root of unity of order d , and W has no repeated zeros.

Thus, in the non-generic case, σ is the set of vertices of a regular d -sided polygon, $d_s = d$, and W is equivalent to $w_\alpha(e^{i\theta d}) = |e^{i\theta d} - 1|^\alpha$.

If σ is not the set of vertices of a regular d -sided polygon (the generic case), then $d_s = 1$, and W equals zero only at $e^{it} = 1$, so that $W \approx w_\alpha$.

3. If d is not a prime number, denote by $d = d_1 > d_2 > \dots > d_N = 1$ all the divisors of d , and let $n_s = d/d_s$ ($s = 1, \dots, N$). As was shown above, if $W(e^{it}) = 0$ for $t \in (0, 2\pi)$, then $t = 2\pi n/d$ for some $n \in J_d$, and there exists a permutation $j \mapsto m(j)$ of J_d such that,

$$\theta_{m(j)} = \theta_j + \frac{2\pi n}{d} \pmod{2\pi}, \quad \forall j \in J_d.$$

Since every permutation can be decomposed into a union of disjoint cycles, and $\theta_j \mapsto \theta_{m(j)}$ is a rotation with the fixed angle t , it follows that all cycles in this decomposition must be of the same length l . Consequently, the length of the cycle must be a divisor of d , i.e., $l = d_k$ for some $k = 1, \dots, N - 1$, and there are $n_k = d/d_k$ disjoint cycles in the decomposition. Geometrically this means that $\sigma = \{e^{i\theta_j}\}_{j \in J_d}$ is the set of vertices of a union of $n_k = d/d_k$ distinct regular d_k -sided polygons. In this case the admissible values of $n \in J_d$ are $n = 0, n_k, 2n_k, \dots, (d_k - 1)n_k$, and the corresponding admissible values of $t = 2\pi j/d_k$ ($j \in J_{d_k}$); i.e., $\{e^{it}\}$ are the roots of unity of order d_k .

Let us denote by d_s the length of the largest cycle l (for all possible values of $n \in J_d$), and set $n_s = d/d_s$. Then σ is a union of the set of vertices of n_s regular d_s -sided polygons, and d_s is the largest such number. In this case, $k \geq s$, so that $d_s \geq d_k$, and $t = 2\pi/d_k$ must be a root of unity of order d_s . From this it follows that d_k is a divisor of d_s , and consequently all admissible values of $n \in J_d$ are

$n = 0, n_s, 2n_s, \dots, (d_s - 1)n_s$. In other words, the zero set of W coincides with the roots of unity of order d_s , i.e., W is equivalent to $w_\alpha(e^{itd_s}) = |e^{itd_s} - 1|^\alpha$ ($s = 1, 2, \dots, N - 1$). In the generic case $d_s = 1$, the points in σ *cannot* be represented as the set of vertices of a union of n_k regular d_k -sided polygons for any $k = 1, 2, \dots, N - 1$. Then W has the only zero at $e^{it} = 1$, and consequently, $W \approx w_\alpha$. This completes the proof of the Lemma. \square

5.2. Weights with singularities of different orders. For the sake of simplicity let us consider a weight with two generic zeros on \mathbb{T} :

$$w = w_\beta w_\alpha^\theta = |e^{it} - 1|^\beta |e^{it} - e^{i\theta}|^\alpha, \quad \theta \neq 2\pi n, \quad \forall n \in \mathbb{Z}.$$

Without loss of generality we assume $0 < \alpha \leq \beta < 1$. In the following theorem we characterize bounded convolution operators $T_\lambda = k \star (\cdot) : L^2(w) \rightarrow L^2(w)$, or equivalently multipliers $\lambda \in \text{Mult}(L^2(w))$, in terms of multipliers involving the weights w_α and w_β , separately. Note that all multiplier algebras discussed below are embedded into $l^\infty(\mathbb{Z})$. The norms in the intersection and the sum of the multiplier spaces are introduced as usual for a Banach couple $(X^{(1)}, X^{(2)})$ ($X^{(1)}, X^{(2)} \subset X$):

$$\|x\|_{X^{(1)} \cap X^{(2)}} = \max(\|x\|_{X^{(1)}}, \|x\|_{X^{(2)}}),$$

$$\|x\|_{X^{(1)} + X^{(2)}} = \inf \left\{ \|x^{(1)}\|_{X^{(1)}} + \|x^{(2)}\|_{X^{(2)}} : x = x^{(1)} + x^{(2)} \right\},$$

where $x^{(1)} \in X^{(1)}, x^{(2)} \in X^{(2)}$.

We will denote the class of bounded Fourier multipliers acting from $L^2(w_1)$ to $L^2(w_2)$ by $\text{Mult}(L^2(w_1) \rightarrow L^2(w_2))$, and in the case $w_1 = w_2 = w$ continue to use the notation $\text{Mult}(L^2(w))$.

It turns out that $\text{Mult}(L^2(w_\beta w_\alpha^\theta))$ can be characterized as the intersection of $\text{Mult}(L^2(w_\alpha))$ and the sum of $\text{Mult}(L^2(w_\beta))$ and the “rotated” multiplier class $\hat{R}_\theta(\text{Mult}(L^2(w_\beta) \rightarrow L^2(w_\alpha)))$:

$$\text{Mult}(L^2(w_\beta w_\alpha^\theta)) = \text{Mult}(L^2(w_\alpha)) \cap \left(\text{Mult}(L^2(w_\beta)) + \hat{R}_\theta(\text{Mult}(L^2(w_\beta) \rightarrow L^2(w_\alpha))) \right).$$

Here $(\hat{R}_\theta \lambda)_j = e^{-ij\theta} \lambda_j$ ($j \in \mathbb{Z}$) is a rotation operator on \mathbb{Z} . Note that the sum of the multiplier spaces above is not a direct sum since

$$\text{Mult}(L^2(w_\beta)) \cap \hat{R}_\theta(\text{Mult}(L^2(w_\beta) \rightarrow L^2(w_\alpha))) = \text{Mult}(L^2(w_\beta) \rightarrow L^2(w_\alpha)).$$

5.3. Theorem. *Let $\lambda = \mathcal{F}k \in l^\infty(\mathbb{Z})$. Let*

$$w = w_\beta w_\alpha^\theta, \quad \theta \neq \pi n, \quad \forall n \in \mathbb{Z},$$

where $0 < \alpha < \beta < 1$. Then the following statements hold.

(1) *The inequality*

$$(5.1) \quad \|k \star f\|_{L^2(w)} \leq C \|f\|_{L^2(w)}$$

holds for all $f \in L^2(w)$ if and only if k can be represented in the form

$$(5.2) \quad k = k^{(1)} + k^{(2)},$$

where k , $k^{(1)}$, and $k^{(2)}$ are pseudo-measures satisfying the following conditions:

$$(5.3) \quad \|k \star f\|_{L^2(w_\alpha)} \leq C_0 \|f\|_{L^2(w_\alpha)},$$

$$(5.4) \quad \|k^{(1)} \star f\|_{L^2(w_\beta)} \leq C_1 \|f\|_{L^2(w_\beta)},$$

$$(5.5) \quad \|(R_{-\theta} k^{(2)}) \star f\|_{L^2(w_\alpha)} \leq C_2 \|f\|_{L^2(w_\beta)},$$

where $(R_{-\theta} k^{(2)})(e^{it}) = k^{(2)}(e^{i(t+\theta)})$.

(2) Condition (5.3) in statement (1) can be replaced with the following condition on $k^{(2)}$:

$$(5.6) \quad \|k^{(2)} \star f\|_{L^2} \leq C_3 \|f\|_{L^2(w_\alpha)}.$$

(3) Decomposition (5.2) can be obtained explicitly as follows:

$$(5.7) \quad k^{(1)} = \eta k, \quad k^{(2)} = (1 - \eta)k,$$

where η is a cut-off function such that $\eta(e^{it}) = 1$ if $|t| < a$, and $\eta(e^{it}) = 0$ outside $|t| < 2a$, for some $0 < a < \pi/4$ so that $4a < |\theta| < \pi$, under the additional assumption that η is in the Wiener algebra on \mathbb{T} , i.e., $\sum_{n \in \mathbb{Z}} |\hat{\eta}(n)| < +\infty$.

(4) In the case $\theta = \pi n$ ($n \in \mathbb{Z}$) inequality (5.1) holds if and only if $k = k^{(1)} + k^{(2)}$, so that (5.4) and (5.5) hold.

5.4. Remarks. 1. In the case $\theta = \pi n$ condition (5.3) in statement (1) is replaced with $\|k \star f\|_{L^2(w_\alpha w_\alpha^\theta)} \leq C_0 \|f\|_{L^2(w_\alpha)}$, which is a consequence of (5.4) and (5.5).

2. Conditions (5.4) and (5.5) automatically imply that $k^{(1)}$, $k^{(2)}$ are pseudo-measures. A direct characterization of the conditions on $k^{(1)}$, $k^{(2)}$ in representation (5.2) in terms of their Fourier coefficients is given below (see Corollary 5.7).

3. It is easy to see that every function $f \in L^2(w_\beta w_\alpha^\theta)$ allows a decomposition $f = f_1 + f_2$ where $f_1 \in L^2(w_\beta)$ and $f_2 \in L^2(w_\alpha^\theta)$, unique up to a summand from the “flat” space $L^2(\mathbb{T}) = L^2(w_\beta) \cap L^2(w_\alpha^\theta)$. We can write this decomposition as follows:

$$L^2(w_\beta w_\alpha^\theta) = L^2(w_\beta) + L^2(w_\alpha^\theta) + L^2(\mathbb{T}).$$

Using this decomposition we can restate the principal claim of Theorem 5.3 as follows: the arrow on the left-hand side of the following diagram (consequently, a multiplier T_λ) represents a bounded operator if and only if $\lambda = \lambda^{(1)} + \lambda^{(2)}$, $T_\lambda = T_{\lambda^{(1)}} + T_{\lambda^{(2)}}$, and all the arrows on the right-hand side represent bounded operators (in the corresponding spaces):

$$\begin{array}{ccc}
L^2(w_\beta w_\alpha^\theta) & = & L^2(w_\beta) + L^2(w_\alpha^\theta) + L^2(\mathbb{T}) \\
\downarrow T_\lambda & & \downarrow T_{\lambda(1)} \searrow T_{\lambda(2)} \searrow T_{\lambda(2)} \\
L^2(w_\beta w_\alpha^\theta) & = & L^2(w_\beta) + L^2(w_\alpha^\theta) + L^2(\mathbb{T})
\end{array}$$

Proof of Theorem 5.3. We start with the following lemma which characterizes the class $\text{Mult}(L^2(w))$ in simpler terms.

5.5. Lemma. (i) Under the assumptions of Theorem 5.3, inequality (5.1) holds, or equivalently $\lambda = \mathcal{F}k \in \text{Mult}(L^2(w))$, if and only if the following pair of inequalities hold:

$$(5.8) \quad \|k \star f\|_{L^2(w_\beta w_\alpha^\theta)} \leq C \|f\|_{L^2(w_\beta)},$$

$$(5.9) \quad \|k \star f\|_{L^2(w_\alpha)} \leq C \|f\|_{L^2(w_\alpha)},$$

provided $\theta \neq \pi n$, $n \in \mathbb{Z}$. (ii) For $\theta = \pi n$, $n \in \mathbb{Z}$, the above statement holds if inequality (5.9) is replaced with

$$(5.10) \quad \|k \star f\|_{L^2(w_\alpha w_\alpha^\theta)} \leq C \|f\|_{L^2(w_\alpha)}.$$

5.6. Remark If $\alpha = \beta \geq 0$ then obviously (5.8) follows from (5.9).

Proof of Lemma 5.5. Let $\tilde{k}(e^{it}) = k(e^{-it})$. Notice that

$$(5.11) \quad \frac{1}{w} = \frac{1}{w_\beta w_\alpha^\theta} \approx w_{-\alpha}^\theta + w_{-\beta}.$$

Then by duality, (5.1) holds if and only if

$$\begin{aligned}
& \|\tilde{k} \star f\|_{L^2(w_{-\beta})}^2 + \|\tilde{k} \star f\|_{L^2(w_{-\alpha}^\theta)}^2 \\
& \leq C \left(\|f\|_{L^2(w_{-\beta})}^2 + \|f\|_{L^2(w_{-\alpha}^\theta)}^2 \right),
\end{aligned}$$

for all trigonometric polynomials f , which are dense in $L^2(w_{-\beta}) \cap L^2(w_{-\alpha}^\theta)$. Consequently, (5.1) is equivalent to the following pair of inequalities:

$$\begin{aligned}
\|\tilde{k} \star f\|_{L^2(w_{-\beta})}^2 & \leq C \left(\|f\|_{L^2(w_{-\beta})}^2 + \|f\|_{L^2(w_{-\alpha}^\theta)}^2 \right), \\
\|\tilde{k} \star f\|_{L^2(w_{-\alpha}^\theta)}^2 & \leq C \left(\|f\|_{L^2(w_{-\beta})}^2 + \|f\|_{L^2(w_{-\alpha}^\theta)}^2 \right).
\end{aligned}$$

Using duality and (5.11) again, we rewrite the preceding inequalities in the equivalent form:

$$(5.12) \quad \|k \star f\|_{L^2(w_\beta w_\alpha^\theta)} \leq C \|f\|_{L^2(w_\beta)},$$

$$(5.13) \quad \|k \star f\|_{L^2(w_\beta w_\alpha^\theta)} \leq C \|f\|_{L^2(w_\alpha^\theta)}.$$

Notice that (5.12) coincides with (5.8). Applying the rotation operator $R_\theta f(e^{it}) = f(e^{i(t-\theta)})$, we see that (5.13) is equivalent to:

$$(5.14) \quad \|k \star f\|_{L^2(w_\alpha w_\beta^{-\theta})} \leq C \|f\|_{L^2(w_\alpha)}.$$

Clearly, $w_\beta w_\alpha^\theta + w_\alpha w_\beta^{-\theta} \approx w_\alpha$ in the generic case ($\theta \neq \pi n$) since $0 < \alpha \leq \beta$. Adding together (5.12) and (5.14) we arrive at the inequality

$$\|k \star f\|_{L^2(w_\alpha)} \leq C \|f\|_{L^2(w_\alpha)},$$

which coincides with (5.9). Moreover, the preceding inequality is stronger than (5.14) since $w_\alpha w_\beta^{-\theta} \leq 2^\beta w_\alpha$. Thus, (5.1) holds if and only if both (5.8) and (5.9) hold.

If $\theta = \pi n$ ($n \in \mathbb{Z}$), then $w_\beta w_\alpha^\theta + w_\alpha w_\beta^{-\theta} \approx w_\alpha w_\beta^\theta$, and $w_\alpha w_\beta^{-\theta} \leq w_\alpha w_\beta^\theta$. Hence (5.14), and consequently (5.1), holds if and only if both (5.8) and (5.10) hold. \square

We now prove the sufficiency part of Theorem 5.3. Suppose that $k = k^{(1)} + k^{(2)}$, where $k^{(1)}$ and $k^{(2)}$ satisfy (5.4) and (5.5) respectively. (The corresponding decomposition of the multiplier λ will be written in the form $\lambda = \lambda^{(1)} + \lambda^{(2)}$ where $\lambda^{(i)} = \mathcal{F}k^{(i)}$.) Notice that (5.5) is equivalent to:

$$(5.15) \quad \|k^{(2)} \star f\|_{L^2(w_\alpha^\theta)} \leq C_2 \|f\|_{L^2(w_\beta)}.$$

Then clearly,

$$\begin{aligned} \|k \star f\|_{L^2(w_\beta w_\alpha^\theta)} &\leq 2^\alpha \|k^{(1)} \star f\|_{L^2(w_\beta)} + 2^\beta \|k^{(2)} \star f\|_{L^2(w_\alpha^\theta)} \\ &\leq (2^\alpha C_1 + 2^\beta C_2) \|f\|_{L^2(w_\beta)}. \end{aligned}$$

This proves (5.8). If $\theta \neq \pi n$ then combining (5.8) with (5.9) yields (5.1) by Lemma 5.5. Similarly, for $\theta = \pi n$, we use (5.8) together with (5.10) to see that (5.1) holds.

Suppose now that $\theta \neq \pi n$, and condition (5.6) is used in place of (5.9), that is, we assume

$$\lambda^{(2)} \in \text{Mult}(L^2(w_\alpha) \longrightarrow L^2).$$

The preceding condition obviously implies

$$(5.16) \quad \lambda^{(2)} \in \text{Mult}(L^2(w_\alpha)).$$

Since $\lambda^{(1)} \in \text{Mult}(L^2(w_\beta))$, it follows that $\lambda^{(1)} \in \text{Mult}(L^2) = l^\infty(\mathbb{Z})$. Hence by interpolation,

$$(5.17) \quad \lambda^{(1)} \in \text{Mult}(L^2(w_\alpha)).$$

Combining (5.16) and (5.17) we obtain (5.9). This completes the proof of the sufficiency part of Theorem 5.3.

To prove the necessity part, suppose (5.1) holds, or equivalently, $\lambda \in \text{Mult}(L^2(w))$. Then (5.8) holds as well by Lemma 3.5. Let η be a cut-off function defined in Theorem 5.3 (3). Consider decomposition (5.7), where $k^{(1)} = \eta k$ and $k^{(2)} = (1 - \eta)k$. Then clearly, $\lambda^{(1)} \in \text{Mult}(L^2(w))$, since

$$k^{(1)} \star f(e^{it}) = \sum_{j \in \mathbb{Z}} \hat{\eta}(j) e^{ijt} (k \star f_j)(e^{it}),$$

where $f_j(e^{it}) = e^{-ijt} f(e^{it})$. Hence,

$$(5.18) \quad \|k^{(1)} \star f\|_{L^2(w)} \leq \sum_{j \in \mathbb{Z}} |\hat{\eta}(j)| \|k \star f_j\|_{L^2(w)} \leq C \|f\|_{L^2(w)} \sum_{j \in \mathbb{Z}} |\hat{\eta}(j)|.$$

Since $\lambda \in \text{Mult}(L^2(w))$, it follows from the preceding inequality that

$$(5.19) \quad \|k^{(2)} \star f\|_{L^2(w)} \leq C \|f\|_{L^2(w)}$$

as well, or equivalently $\lambda^{(2)} \in \text{Mult}(L^2(w))$.

Next, we estimate

$$\|k^{(1)} \star f\|_{L^2(w_\beta)}^2 = \int_{|t| < 3a} |k^{(1)} \star f|^2 w_\beta dt + \int_{|t| \geq 3a} |k^{(1)} \star f|^2 w_\beta dt = I + II.$$

Since $w(e^{it}) \approx w_\beta(e^{it})$ for $|t| < 3a$, we deduce from (5.8) and (5.18):

$$I \leq \|k^{(1)} \star f\|_{L^2(w)}^2 \leq C \|k_1 \star f\|_{L^2(w)}^2 \leq C_1 \|f\|_{L^2(w_\beta)}^2.$$

To estimate the second term, notice that $k^{(1)}(e^{i(t-\tau)})$ is supported in $|t - \tau| \leq 2a$. Hence, for $|t| \geq 3a$, we have $|\tau| \geq a$, so that we may assume that $f(e^{i\tau})$ is supported in $|\tau| \geq a$. In other words, we may replace f in II with $f\chi_{|\tau| \geq a}$. Moreover, for $|t| \geq 3a$, $w_\beta(e^{it}) \approx 1$. It follows,

$$II \leq \|k^{(1)} \star (f\chi_{|\tau| \geq a})\|_{L^2}^2 \leq \|\lambda^{(1)}\|_{l^\infty}^2 \|f\chi_{|\tau| \geq a}\|_{L^2}^2 \leq C \|f\|_{L^2(w_\beta)}^2.$$

Combining the preceding estimates, we see that $\lambda^{(1)} \in \text{Mult}(L^2(w_\beta))$.

Similarly, for $k^{(2)} = (1 - \eta)k$ and $(R_{-\theta}k^{(2)})(e^{i\tau}) = (1 - \eta(e^{i(\tau+\theta)}))k(e^{i(\tau+\theta)})$, we obtain:

$$\begin{aligned} \|(R_{-\theta}k^{(2)}) \star f\|_{L^2(w_\alpha)}^2 &= \|k^{(2)} \star f\|_{L^2(w_\alpha^\theta)}^2 \\ &= \int_{|t| < \frac{a}{2}} |k^{(2)} \star f|^2 w_\alpha^\theta dt + \int_{|t| \geq \frac{a}{2}} |k^{(2)} \star f|^2 w_\alpha^\theta dt \\ &= III + IV. \end{aligned}$$

We first estimate III . Since $k^{(2)}(e^{i(t-\tau)})$ is supported in $|t - \tau| \geq a$ and $|t| < \frac{a}{2}$, it follows that $f(e^{i\tau})$ can be replaced with $f\chi_{|\tau| > \frac{a}{2}}$. Hence,

$$\begin{aligned} III &\leq \|k^{(2)} \star f\chi_{|\tau| > \frac{a}{2}}\|_{L^2(w_\alpha^\theta)}^2 \leq \|k^{(2)} \star f\chi_{|\tau| > \frac{a}{2}}\|_{L^2}^2 \\ &\leq \|k^{(2)}\|_{l^\infty} \|f\chi_{|\tau| > \frac{a}{2}}\|_{L^2}^2 \leq C \|k^{(2)}\|_{l^\infty} \|f\|_{L^2(w_\beta)}^2. \end{aligned}$$

To estimate IV , we notice that, for $|t| \geq \frac{a}{2}$,

$$w_\alpha^\theta(e^{it}) \approx w_\alpha^\theta(e^{it}) w_\beta(e^{it}) = w(e^{it}).$$

Consequently, using (5.19), we estimate:

$$IV \leq C_1 \|k^{(2)} \star f\|_{L^2(w)}^2 \leq C_2 \|k \star f\|_{L^2(w)}^2 \leq C_3 \|f\|_{L^2(w)}^2.$$

Combining these estimates, we deduce

$$\lambda^{(2)} \in \text{Mult}(L^2(w_\beta) \longrightarrow L^2(w_\alpha^\theta)),$$

which is equivalent to (5.5):

$$(5.20) \quad \|k^{(2)} \star f\|_{L^2(w_\alpha^\theta)} \leq C \|f\|_{L^2(w_\beta)}.$$

This completes the proof in the case $\theta = \pi n$.

The above estimates remain true if $\theta \neq \pi n$. Additionally, it follows from Lemma 5.2 that (5.9) holds. Using (5.18), (5.19) with w_α in place of w we deduce that (5.9) holds with $k^{(2)}$ in place of k :

$$(5.21) \quad \|k^{(2)} \star f\|_{L^2(w_\alpha)} \leq C \|f\|_{L^2(w_\alpha)}.$$

Adding together (5.20) and (5.21) we obtain:

$$\|k^{(2)} \star f\|_{L^2} \leq C \|f\|_{L^2(w_\alpha)}.$$

This proves (5.6), and completes the proof of the necessity part of Theorem 5.3 in the generic case $\theta \neq \pi n$. \square

Combining Theorem 5.3 with Theorem 4.11 we obtain the following characterization of multipliers.

5.7. Corollary. *Under the assumptions of Theorem 5.3, inequality (5.1) holds if and only if $k = k^{(1)} + k^{(2)}$, where $\lambda^{(i)} = \mathcal{F}k^{(i)} \in l^\infty(\mathbb{Z})$ ($i = 1, 2$), and for every finite $J \subset \mathbb{Z}$, the following conditions hold:*

$$(5.22) \quad \sum_{j \in J} \sum_{m \in \mathbb{Z}} \frac{|\lambda_j^{(1)} - \lambda_m^{(1)}|^2}{(|j - m| + 1)^{1+\beta}} \leq C \text{Cap}_\beta(J),$$

$$(5.23) \quad \sum_{j \in J} \sum_{m \in \mathbb{Z}} \frac{|e^{ij\theta} \lambda_j^{(2)} - e^{im\theta} \lambda_m^{(2)}|^2}{(|j - m| + 1)^{1+\alpha}} \leq C \text{Cap}_\beta(J),$$

$$(5.24) \quad \sum_{j \in J} |\lambda_j^{(2)}|^2 \leq C \text{Cap}_\gamma(J),$$

where $\gamma = \max(\alpha, \beta - \alpha)$ and C does not depend on J , provided $\theta \neq \pi n$, $n \in \mathbb{Z}$.

In the case $\theta = \pi n$, (5.1) holds if and only if $k = k^{(1)} + k^{(2)}$ so that $\lambda^{(i)} \in l^\infty(\mathbb{Z})$ ($i = 1, 2$), and (5.22), (5.23) hold.

Remark. In condition (5.24) it suffices to let $\gamma = \alpha$. However, the proof of this assertion is complicated, and we do not present it here. (It requires discrete analogues of multiplier estimates obtained earlier by the second author in the continuous case; see Sec. 3.2.10 in [MSh2009].)

The following example demonstrates that conditions (5.22)–(5.24) are essential, and in a sense cannot be relaxed. Moreover, the inequality

$$(5.25) \quad \|k \star f\|_{L^2(w_\beta)} \leq C \|f\|_{L^2(w_\beta)},$$

is only sufficient, but not necessary for (5.1) in the case $\alpha < \beta$, contrary to the case $\alpha = \beta$. In other words, we cannot let $k^{(2)} = 0$ in decomposition (5.2). On the other hand, condition (5.3) is only necessary, but not sufficient for (5.1).

5.8. Example. Suppose $w = w_\beta w_\alpha^\theta$ where $0 < \alpha < \beta < 1$ and $\theta \neq \pi n$ ($n \in \mathbb{Z}$). For $\gamma = \max(\alpha, \beta - \alpha)$, we pick $\delta > 0$ so that $\frac{\gamma}{2} < \delta < \frac{\beta}{2}$. Let $\lambda = (\lambda_j)$, where $\lambda_j = \frac{e^{-ij\theta}}{(|j|+1)^\delta}$ ($j \in \mathbb{Z}$). Then conditions (5.23)–(5.24) hold for $k = k^{(2)}$, $k^{(1)} = 0$, but (5.22) fails for $k = k^{(1)}$, $k^{(2)} = 0$. In other words, $\lambda \in \text{Mult}(L^2(w))$, but $\lambda \notin \text{Mult}(L^2(w_\beta))$. Moreover, $\bar{\lambda} = (\bar{\lambda}_j) \notin \text{Mult}(L^2(w))$. If $\delta < \frac{\gamma}{2}$, then $\lambda \notin \text{Mult}(L^2(w))$.

The claims in Example 5.8 follow from the well-known fact that if $\Lambda = (\Lambda_j)$, where

$$\Lambda_j = \frac{1}{(|j|+1)^\delta}, \quad j \in \mathbb{Z}, \quad 0 < \delta < 1,$$

then $\Lambda \in \text{Mult}(L^2(w_\beta) \rightarrow L^2(w_\alpha))$ for $0 \leq \alpha < \beta < 1$ if and only if $0 < \delta \leq \frac{\beta-\alpha}{2}$. Note that in this example $\lambda \in \text{Mult}(L^2(w))$, but $\bar{\lambda} \notin \text{Mult}(L^2(w))$, since otherwise by interpolation the Fourier multiplier $T_\Lambda: L^2(w_\beta) \rightarrow L^2(w_\alpha)$ would be bounded, which fails for $\frac{\beta-\alpha}{2} < \delta$.

The next theorem shows that, for weights with a finite number of power-like singularities, $\text{Mult}(L^2(w))$ has the Spectral Localization Property (SLP). As above, for the sake of simplicity, we consider weights with two generic zeros,

$$w = w_\alpha^\theta w_\beta, \quad \theta \neq \pi n, \quad \forall n \in \mathbb{Z},$$

where $0 < \alpha \leq \beta < 1$. (It is easy to see that in the case $\theta = \pi$ the SLP holds as well.)

5.9. Theorem. *Under the hypotheses of Theorem 5.3, suppose $\lambda \in \text{Mult}(L^2(w))$, so that $\lambda = \mathcal{F}k$, where k is a pseudo-measure on \mathbb{T} such that the inequality*

$$(5.26) \quad \|k \star f\|_{L^2(w)} \leq C \|f\|_{L^2(w)}$$

holds for all $f \in L^2(w)$. If $\inf_{j \in \mathbb{Z}} |\lambda_j| > 0$, then $1/\lambda \in \text{Mult}(L^2(w))$.

Proof. Suppose $\lambda \in \text{Mult}(L^2(w))$ and $\inf_{j \in \mathbb{Z}} |\lambda_j| = \delta > 0$. By Theorem 5.3, $\lambda = \lambda^{(1)} + \lambda^{(2)}$, where the following three conditions hold:

$$(5.27) \quad \lambda \in \text{Mult}(L^2(w_\alpha)),$$

$$(5.28) \quad \lambda^{(1)} \in \text{Mult}(L^2(w_\beta)),$$

$$(5.29) \quad \lambda^{(2)} \in \text{Mult}(L^2(w_\beta) \longrightarrow L^2(w_\alpha^\theta)) \bigcap \text{Mult}(L^2(w_\alpha) \longrightarrow L^2).$$

We will also need the following relations which follow from (5.27) and (5.29) respectively by applying the rotation operator \hat{R}_θ :

$$(5.30) \quad \lambda \in \text{Mult}(L^2(w_\alpha^\theta)),$$

$$(5.31) \quad \lambda^{(2)} \in \text{Mult}(L^2(w_\alpha^\theta) \longrightarrow L^2).$$

We first prove Theorem 5.9 under the additional assumption

$$(5.32) \quad \inf_{j \in \mathbb{Z}} |\lambda_j^{(1)}| > 0.$$

Then

$$(5.33) \quad \frac{1}{\lambda} = \frac{1}{\lambda^{(1)}} + \frac{-\lambda^{(2)}}{\lambda \lambda^{(1)}}.$$

Using assumption (5.32) and the SLP for the weight w_β , we deduce from (5.28):

$$(5.34) \quad \frac{1}{\lambda^{(1)}} \in \text{Mult}(L^2(w_\beta)).$$

Since $1/\lambda^{(1)} \in \text{Mult}(L^2) = l^\infty$, and $\alpha \leq \beta$, using interpolation we see that $1/\lambda^{(1)} \in \text{Mult}(L^2(w_\alpha))$. Applying the rotation operator R_θ , we obtain

$$(5.35) \quad \frac{1}{\lambda^{(1)}} \in \text{Mult}(L^2(w_\alpha^\theta)).$$

By (5.30) and the SLP for the weight w_α^θ ,

$$(5.36) \quad \frac{1}{\lambda} \in \text{Mult}(L^2(w_\alpha^\theta)).$$

By (5.29), $\lambda^{(2)} \in \text{Mult}(L^2(w_\beta) \longrightarrow L^2(w_\alpha^\theta))$. Consequently, as a product of three multipliers,

$$\frac{\lambda^{(2)}}{\lambda \lambda^{(1)}} \in \text{Mult}(L^2(w_\beta) \longrightarrow L^2(w_\alpha^\theta)).$$

From (5.31) and (5.35), (5.36) it follows that

$$\frac{\lambda^{(2)}}{\lambda \lambda^{(1)}} \in \text{Mult}(L^2(w_\alpha^\theta) \longrightarrow L^2)$$

as well. This proves that decomposition (5.33) for $1/\lambda$ is of the same type as for λ in Theorem 5.3. Thus, by the sufficiency part of Theorem 5.9, we conclude that $1/\lambda \in \text{Mult}(L^2(w))$.

We now demonstrate how to remove the additional assumption (5.32) used above. Suppose

$$(5.37) \quad \delta = \inf_{j \in \mathbb{Z}} |\lambda_j| > 0.$$

Let $k = k^{(1)} + k^{(2)}$ be decomposition (5.2) obtained in Theorem 5.3. Let

$$Z_1 = \left\{ j \in \mathbb{Z} : |\lambda_j^{(1)}| \geq \frac{\delta}{2} \right\}, \quad Z_2 = \mathbb{Z} \setminus Z_1.$$

Then, obviously,

$$(5.38) \quad \inf_{j \in Z_2} |\lambda_j^{(2)}| \geq \frac{\delta}{2}.$$

We claim that the following inequality holds for every $f \in L^2(w_\beta)$:

$$(5.39) \quad \sum_{j \in Z_2} |\hat{f}(j)|^2 \leq C \|f\|_{L^2(w_\beta)}^2.$$

In other words, the set Z_2 is quite meager. Indeed, since

$$\lambda^{(2)} \in \text{Mult}(L^2(w_\beta) \longrightarrow L^2(w_\alpha)) \bigcap \text{Mult}(L^2(w_\alpha^\theta) \longrightarrow L^2),$$

we obtain for every $f \in L^2(w_\beta)$:

$$\begin{aligned} \sum_{j \in \mathbb{Z}} |\lambda_j^{(2)}|^4 |\hat{f}(j)|^2 &= \|k^{(2)} \star (k^{(2)} \star f)\|_{L^2}^2 \\ &\leq C \|k^{(2)} \star f\|_{L^2(w_\alpha^\theta)}^2 \leq C \|f\|_{L^2(w_\beta)}^2. \end{aligned}$$

On the other hand, by (5.38),

$$\begin{aligned} \sum_{j \in Z_2} |\hat{f}(j)|^2 &\leq \frac{16}{\delta^4} \sum_{j \in Z_2} |\lambda_j^{(2)}|^4 |\hat{f}(j)|^2 \\ &\leq \frac{16}{\delta^4} \sum_{j \in \mathbb{Z}} |\lambda_j^{(2)}|^4 |\hat{f}(j)|^2. \end{aligned}$$

Combining the preceding estimates, we prove (5.39).

From (5.39) we deduce:

$$\sum_{j \in Z_2} |\lambda_j^{(2)}|^2 |\hat{f}(j)|^2 \leq \|\lambda^{(2)}\|_{l^\infty}^2 \sum_{j \in Z_2} |\hat{f}(j)|^2 \leq C \|f\|_{L^2(w_\beta)}^2.$$

This yields:

$$(5.40) \quad \lambda^{(2)} \chi_{Z_2} \in \text{Mult}(L^2(w_\beta) \longrightarrow L^2).$$

We can now adjust decomposition (5.2): $k = k^{(3)} + k^{(4)}$, where

$$\lambda^{(3)} = \lambda^{(1)} + \lambda^{(2)} \chi_{Z_2}, \quad \lambda^{(4)} = \lambda^{(2)} - \lambda^{(2)} \chi_{Z_2}.$$

Clearly, by (5.37) and (5.38),

$$(5.41) \quad \inf_{j \in \mathbb{Z}} |\lambda_j^{(3)}| \geq \frac{\delta}{2}.$$

Moreover, (5.40) yields that the required conditions still hold for the new components $\lambda^{(3)}$ and $\lambda^{(4)}$:

$$\begin{aligned} \lambda^{(3)} &\in \text{Mult}(L^2(w_\beta)), \\ \lambda^{(4)} &\in \text{Mult}(L^2(w_\beta) \longrightarrow L^2(w_\alpha^\theta)), \\ \lambda^{(4)} &\in \text{Mult}(L^2(w_\alpha) \longrightarrow L^2). \end{aligned}$$

Thus, assumption (5.32) is redundant in the general case. This proves that $1/\lambda \in \text{Mult}(L^2(w))$. \square

5.10. Remark. The same argument as in Theorem 5.3 works in the non-generic case $\theta = \pi$ as well. For instance, for $\alpha = \beta$, i.e., $w = w_\beta w_\beta^\pi$, or $w = w_{-\beta} w_{-\beta}^\pi$ ($0 < \beta < 1$), it follows that $\lambda \in \text{Mult}(L^2(w))$ if and only if

$$\lambda = \lambda^{(1)} + \hat{R}_\pi \lambda^{(2)},$$

where $\lambda^{(1)}, \lambda^{(2)} \in \text{Mult}(L^2(w_\beta))$, and $\hat{R}_\pi \lambda = ((-1)^j \lambda_j)$ ($j \in \mathbb{Z}$) is the corresponding rotation operator. An equivalent direct characterization of $\text{Mult}(L^2(w))$ is given in Sec. 5.1.

5.11. Remark. More generally, suppose $w = w_{-\beta} \star \nu$ ($0 < \beta < 1$), where $\nu = \sum_{j=0}^{d-1} c_j \delta_{\zeta^j}$ is a finite discrete measure ($c_j > 0$), and $\{\zeta^j\}$ are the roots of unity of order d . An explicit description of multipliers $\text{Mult}(L^2(w))$ is given in Sec. 5.1. There is an alternative “sliced” decomposition: $\lambda \in \text{Mult}(L^2(w))$ if and only if

$$\lambda = \sum_{j=0}^{d-1} \hat{R}_{\zeta^j} \lambda^{(j)},$$

where $\hat{R}_\zeta \lambda = (\bar{\zeta}^j \lambda_j)$ is the corresponding rotation operator, and each $\lambda^{(j)} \in \text{Mult}(L^2(w_\beta))$, $j = 1, 2, \dots, d-1$. This can be proved by means of a decomposition similar to that used in the proof of Theorem 5.3 with smooth cut-off functions η_j , $0 \leq j \leq d-1$.

The case of infinitely many singularities, which is briefly discussed below, is quite different.

5.12. Infinite superposition of LKS singularities: the hidden spectrum.

Continuing Remark 5.11, one can consider the case of an infinite combination of LKS singularities (in a dual form), say (for the sake of simplicity) of the same order, as follows: $w = w_{-\beta} \star \nu$, where $0 < \beta < 1$, and

$$\nu = \sum_{k \geq 0} c_k \delta_{\zeta^k}, \quad \text{where } c_k > 0, \quad \sum_{k \geq 0} c_k < \infty, \quad \sup_{k \geq 0} \frac{c_k}{c_{k+1}} < \infty,$$

with $\zeta \in \mathbb{T}$ such that $\zeta^k \neq 1$ ($\forall k \in \mathbb{Z}$). In this case, at the moment, we can only conjecture (but not prove) that there is an analogue of a “sliced” decomposition from Remark 5.11 for a multiplier $T_\lambda \in \text{Mult}(L^2(w_{-\beta} \star \nu))$. We recall, however, that the situation can be more complicated: in [Nik2009] it is shown that in this case there exists a “hidden spectrum,” i.e. the SLP does not hold.

In fact, we believe that the reason why the “hidden spectrum” appears lies in a kind of holomorphic extension of multipliers $n \mapsto \lambda_n$ ($n \in \mathbb{Z}$) of the space $L^2(\nu)$, followed with a “sliced decomposition” mentioned above. The latter property is still a conjecture, but the former one (namely, the holomorphic nature of $\text{Mult}(L^2(\nu))$) is confirmed by the following claim, for which we need a bit of

notation. In order to distinguish Fourier multipliers of $L^2(\nu)$ from yet another (pointwise) holomorphic multipliers appearing in the next theorem, we temporarily change the notation for the former adding a subscript “ F ” (for Fourier):

$$\text{Mult}_F(L^2(\mathbb{T}, \nu)) := \text{Mult}(L^2(\mathbb{T}, \nu)).$$

For $c = (c_k)_{k \geq 0}$, let $1/c = (1/c_k)_{k \geq 0}$, and denote by $l_a^2(1/c)$ the Hilbert space of functions f holomorphic on the unit disc \mathbb{D} such that

$$\|f\|^2 = \sum_{k \geq 0} |\hat{f}(k)|^2 \frac{1}{c_k} < \infty.$$

Let $\text{Mult}(l_a^2(1/c))$ stand for (standard, pointwise) multipliers of $l_a^2(1/c)$:

$$\varphi \in \text{Mult}(l_a^2(1/c)) \Leftrightarrow \{f \in l_a^2(1/c) \Rightarrow \varphi f \in l_a^2(1/c)\}.$$

In particular, if $l_a^2(1/c)$ is an algebra (for instance, when $c_k = 1/(k+1)^{1+\epsilon}$, $\epsilon > 0$), then

$$\text{Mult}(l_a^2(1/c)) = \text{mult}(l_a^2(1/c)) = l_a^2(1/c),$$

where $\text{mult}(l_a^2(1/c))$ stands for the closure of polynomials in the norm $\|\cdot\|_{\text{Mult}(l_a^2(1/c))}$. Note that we always have $\text{Mult}(l_a^2(1/c)) \subset l_a^2(1/c) \subset l_a^1$ (the Wiener algebra).

5.13. Theorem. *Under the above assumptions on ν , $c = (c_k)$ and ζ ,*

$$\text{Mult}_F(L^2(\mathbb{T}, \nu)) = \{\lambda_n = \varphi(\zeta^n) (\forall n \in \mathbb{Z}) : \varphi \in \text{Mult}(l_a^2(1/c))\}.$$

Moreover, the “visible spectrum” of a multiplier $\lambda = (\lambda_n)_{n \in \mathbb{Z}} = (\varphi(\zeta^n))_{n \in \mathbb{Z}}$ is a continuous curve

$$\text{clos}\{\varphi(\zeta^n) : n \in \mathbb{Z}\} = \varphi(\mathbb{T}),$$

but the entire spectrum is the φ -image of the closed disc $\overline{\mathbb{D}}$:

$$\sigma(T_\lambda) = \varphi(\overline{\mathbb{D}}),$$

and, at least for $\varphi \in \text{mult}(l_a^2(1/c))$, every point $z \in \varphi(\mathbb{D}) \setminus \varphi(\mathbb{T})$ is a Fredholm point of T_λ so that

$$\text{ind}(T_\lambda - zI) = \dim \text{Ker}(T_\lambda - zI) = \text{wind}(\varphi - z).$$

Proof. Notice that

$$L^2(\mathbb{T}, \nu) = \{(f(\zeta^k))_{k \geq 0} : \int_{\mathbb{T}} |f|^2 d\nu = \sum_{k \geq 0} |f(\zeta^k)|^2 c_k < \infty\} = l_a^2(c).$$

(We use a natural identification, $(a_k) = (f(\zeta^k)) \mapsto \sum_{k \geq 0} a_k z^k$.) By the hypothesis the backward shift $S^*(f(\zeta^k))_{k \geq 0} = (f(\zeta^{k+1}))_{k \geq 0}$ is a bounded operator on $L^2(\mathbb{T}, \nu) = l_a^2(c)$. But $S^* = T_{\{\zeta^n\}}$ is in $\text{Mult}_F(L^2(\mathbb{T}, \nu))$, since

$$S^* z^n = S^*((\zeta^k)^n)_{k \geq 0} = ((\zeta^{k+1})^n)_{k \geq 0} = \zeta^n ((\zeta^k)^n)_{k \geq 0} = \zeta^n z^n, \quad \forall n \in \mathbb{Z}.$$

Consequently, any multiplier operator $T_\lambda \in \text{Mult}_F(L^2(\mathbb{T}, \nu))$ commutes with S^* , and hence T_λ^* commutes with the shift S on the dual space $l_a^2(1/c)$. So,

$T_\lambda^* = \varphi(S) \in \text{Mult}(l_a^2(1/c))$. For every $F \in l_a^2(1/c)$, we have (under the bilinear duality $\langle F, G \rangle = \sum_{k \geq 0} \hat{F}(k) \hat{G}(k)$):

$$\begin{aligned} \langle T_\lambda z^n, F \rangle &= \langle ((\zeta^k)^n)_{k \geq 0}, T_\lambda^* F \rangle = \langle ((\zeta^k)^n)_{k \geq 0}, \varphi F \rangle = \varphi(\zeta^n) F(\zeta^n) \\ &= \varphi(\zeta^n) \langle ((\zeta^k)^n)_{k \geq 0}, F \rangle = \varphi(\zeta^n) \langle z^n, F \rangle. \end{aligned}$$

Hence $T_\lambda z^n = \varphi(\zeta^n) z^n$ for every $n \in \mathbb{Z}$, i.e., $T_\lambda = \varphi(S)^*$.

Clearly, the converse is also true, i.e., $\varphi(\zeta^n)_{n \in \mathbb{Z}}$ is in $\text{Mult}_F(L^2(\mathbb{T}, \nu))$ for every $\varphi \in \text{Mult}(l_a^2(1/c))$.

The spectral nature of the adjoint operator $T_\lambda = \varphi(S)^*$ related to a multiplier $\varphi \in \text{Mult}(l_a^2(1/c))$ is well known (see [Nik1986]). \square

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